# The Precise Definition of the Definite Integral

## Introduction

**The Area Under a Curve**

The **area** $A$ of the region that lies under the graph of the continuous function $f$ is the limit of the sum of the areas of approximating rectangles

$$A = \lim_{n\to\infty} [f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x]$$

## Sigma Notation

We often use **sigma notation** to write sums with many terms more compactly. For instance:

$$\sum_{i=1}^{n} f(x_i^*)\Delta x = f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x$$

**Example 1.**

Express the following sums using sigma notation:

- $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8$

- $1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2$

- $1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3 + 8^3$

**Properties of Sigma Notation**

- $\sum_{i=1}^{n} c = nc$
- $\sum_{i=1}^{n} ca_i = c\left(\sum_{i=1}^{n} a_i\right)$
- $\sum_{i=1}^{n} (a_i + b_i) = \left(\sum_{i=1}^{n} a_i\right) + \left(\sum_{i=1}^{n} b_i\right)$
- $\sum_{i=1}^{n} (a_i - b_i) = \left(\sum_{i=1}^{n} a_i\right) - \left(\sum_{i=1}^{n} b_i\right)$

**Three Useful Sums**

- $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
- $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
- $\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$

**Example 2.**

Find the following sums:

- $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8$
- $1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2$
- $1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3 + 8^3$

## The Definite Integral

**The Area Under a Curve**

The **area** $A$ of the region that lies under the graph of the continuous function $f$ is the limit of the sum of the areas of approximating rectangles

$$A = \lim_{n\to\infty} [f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x]$$

**The Definite Integral**

Let $f$ be a function defined for $a \le x \le b$. The **definite integral of $f$ from $a$ to $b$** is:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

provided that this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that $f$ is **integrable** on $[a, b]$.

**What Functions are Integrable?**

If $f$ is continuous on $[a, b]$, or if $f$ has only a finite number of jump discontinuities, then $f$ is integrable on $[a, b]$; that is, the definite integral $\int_a^b f(x)\,dx$ exists.

## How to Compute the Definite Integral of an Integrable Function

**Using Right Endpoints**

If $f$ is integrable on $[a, b]$, then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

where

$$\Delta x = \frac{b-a}{n} \qquad \text{and} \qquad x_i = a + i\Delta x$$

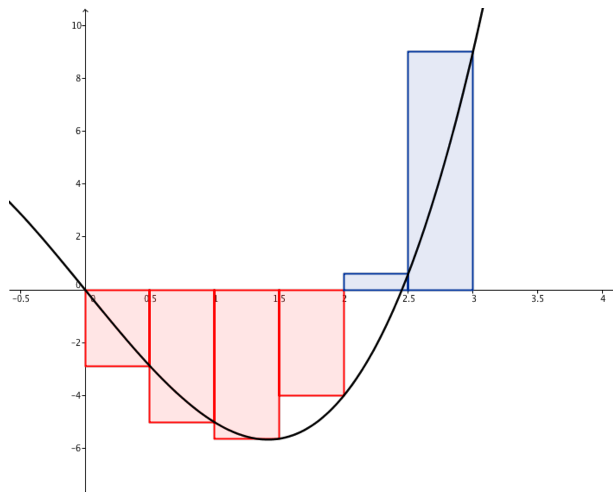
**Example 3.**

Express the following limit as an integral on the interval $[0, \pi]$:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(x_i^3 + x_i \sin x_i\right) \Delta x$$

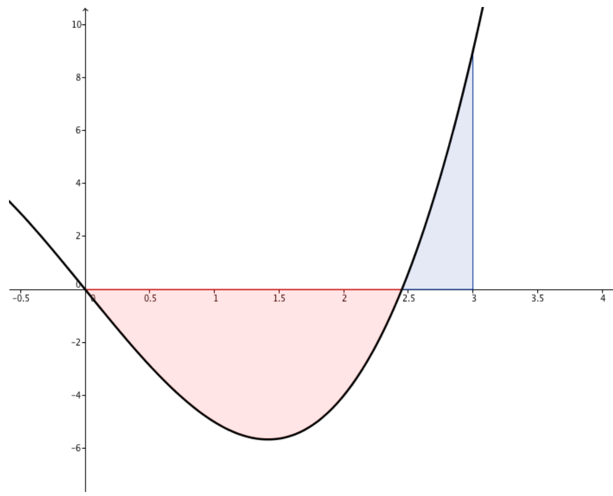
**Example 4.**

Evaluate the Riemann sum for $f(x) = x^3 - 6x$, taking the sample points to be right endpoints and $a = 0$, $b = 3$, and $n = 6$.



**Example 5.**

Evaluate $\int_0^3 \left(x^3 - 6x\right)\, dx$.



## The Midpoint Rule

**The Midpoint Rule**

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(\overline{x_i})\Delta x = [f(\overline{x_1}) + f(\overline{x_2}) + \cdots + f(\overline{x_n})]\,\Delta x$$

where

$$\Delta x = \frac{b-a}{n} \qquad \text{and} \qquad \overline{x_i} = a + \left(i - \frac{1}{2}\right)\Delta x$$

**Example 6.**

Use the Midpoint Rule with $n = 5$ to approximate $\displaystyle\int_1^2 \frac{1}{x}\,dx$.