Continuous Functions

Continuity

Given a function f and a number a, we now consider both:

1. the limit of f as x approaches a
$$\lim_{x \to a} f(x)$$
2. and the value of the function f at $x = a$.
$$f(a)$$

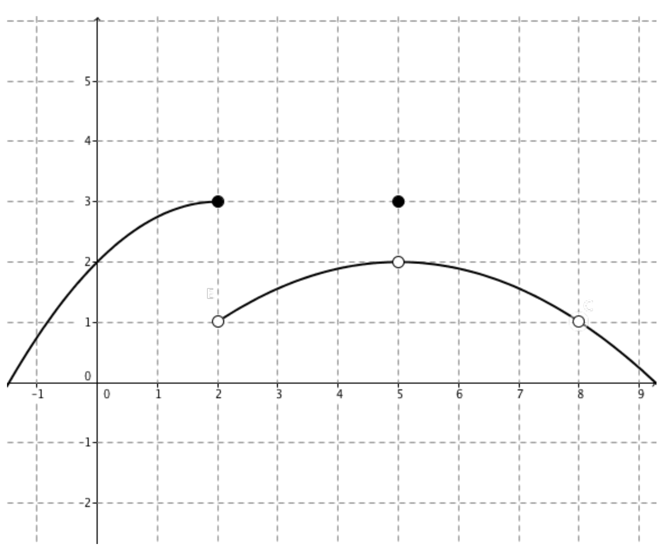
Continuous

A function f is **continuous at a number** a if
$$\lim_{x \to a} f(x) = f(a)$$
otherwise f is **discontinuous at** a.

Comment

Example 1.

Find the numbers where f is discontinuous.



Example 2.

Where are each of the following functions discontinuous?

1. $f(x) = \dfrac{x^2 - x - 2}{x - 2}$

2. $f(x) = \begin{cases} \frac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

3. $f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$

4. $f(x) = \begin{cases} x^2 + 1 & \text{if } x < 0 \\ x^2 & \text{if } x > 0 \end{cases}$

Continuity on an Interval

One-Sided Continuity

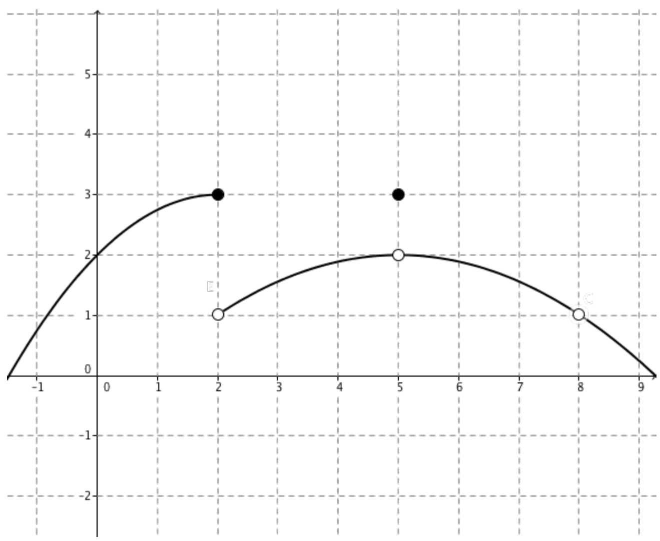
A function f is

continuous from the right at a number a if $\lim_{x \to a^+} f(x) = f(a)$

continuous from the left at a number a if $\lim_{x \to a^-} f(x) = f(a)$

Example 3.

Further discuss the continuity at $x = 2$ and $x = 5$.



Continuous on an Interval

A function f is **continuous on an interval** if it is continuous at every number in the interval.

Example 4

Show that the function $f(x) = 1 - \sqrt{1 - x^2}$ is continuous on the interval $[-1, 1]$

Continuity and Combinations of Functions

Theorem

If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:

1. $f + g$
2. $f - g$
3. cf
4. fg
5. $\dfrac{f}{g}$ if $g(a) \neq 0$

Theorem

The following types of functions are continuous at every number in their domains:

Polynomial		
Rational Functions		
Root Functions	even root	
	odd root	
Trigonometric Functions	$\sin x$	
	$\cos x$	
	$\tan x$	
Inverse Trigonometric Functions	$\arcsin x$	
	$\arccos x$	
	$\arctan x$	
Exponential Functions		
Logarithmic Functions		

Continuity generalizes the **Direct Substitution Method** of evaluating limits that we used with polynomials and rational functions.

Generalized Direct Substitution Method

If a function f is continuous at a then

$$\lim_{x \to a} f(x) = f(a)$$

Example 5a.

Where is the function $f(x) = \dfrac{x^3 + 2x^2 - 1}{5 - 3x}$ continuous?

Example 5b. (Use Continuity to Evaluate a Limit)

Evaluate $\lim\limits_{x \to 1} \dfrac{x^3 + 2x^2 - 1}{5 - 3x}$

Example 6a.

Where is the function $f(x) = \dfrac{\ln x + \arctan x}{x^2 - 1}$ continuous?

Example 6b. (Use Continuity to Evaluate a Limit)

Evaluate $\displaystyle\lim_{x \to e} \frac{\ln x + \arctan x}{x^2 - 1}$

Continuity and Composite Functions

Composition of Functions

Given two functions f and g, the **composite function** $f \circ g$ is defined by

$$(f \circ g)(x) = f(g(x))$$

Example 7.

If $f(x) = x^2$ and $g(x) = x - 3$, find the composite functions $f \circ g$ and $g \circ f$.

Theorem - Outer Function is Continuous

If f is continuous at b and $\lim_{x \to a} g(x) = b$, then $\lim_{x \to a} f(g(x)) = f(b)$. In other words,

$$\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x)).$$

Example 8.

Evaluate $\lim_{x \to 1} \arcsin\left(\frac{1 - \sqrt{x}}{1 - x}\right)$.

Theorem - Both Functions Continuous

If g is continuous at a and f is continuous at $g(a)$ then the composite function $f \circ g$ is continuous at a.

Example 9a.

Where are the following functions continuous?

1. $h(x) = \sin(x^2 + 3x + \pi/2)$
2. $F(x) = \ln(1 + \cos x)$

Example 9b. (Use Continuity to Evaluate a Limit)

Evaluate $\lim_{x \to 0} h(x)$ and $\lim_{x \to 0} F(x)$