Indeterminate Forms and L'Hospital's Rule

Review: Limits Involving Quotients

As we have seen one of the recurring problems in this course is finding the limit of the quotient of two functions. Recall:

(Quotient Law) $$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \qquad \text{if } \lim_{x\to a} g(x) \neq 0$$

- $\lim_{x\to a} g(x) \neq 0$

- $\lim_{x\to a} g(x) = 0$

Indeterminate Quotients

Indeterminate Form 0/0

The limit $\lim_{x\to c} \frac{f(x)}{g(x)}$ is a **0/0 indeterminate form** if both

- $\lim_{x\to c} f(x) = 0$ and
- $\lim_{x\to c} g(x) = 0$

Indeterminate Form ∞/∞

The limit $\lim_{x\to c} \frac{f(x)}{g(x)}$ is an ∞/∞ **indeterminate form** if both

- $\lim_{x\to c} f(x) = +\infty \quad \text{or} -\infty \qquad \text{and}$
- $\lim_{x\to c} g(x) = +\infty \quad \text{or} -\infty$

L'Hospital's Rule

If we have an indeterminate form of type 0/0 or ∞/∞, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$$

provided that the second limit exists or is ∞ or $-\infty$.

Example 1.

Evaluate

$$\lim_{x \to 1} \frac{\ln x}{x - 1}$$

Example 2. [L'Hospital's Rule Is Not Applicable]

Evaluate

$$\lim_{x \to \pi^-} \frac{\sin x}{1 - \cos x}$$

Example 3.

Evaluate

$$\lim_{x \to \infty} \frac{e^x}{x^2}$$

Example 4.

Evaluate

$$\lim_{x \to \infty} \frac{\ln x}{\sqrt[3]{x}}$$

Example 5. [L'Hospital's Rule May Be Used More Than Once]

Evaluate

$$\lim_{x \to 0} \frac{\tan x - x}{x^3}$$

Indeterminate Products

Indeterminate Form $0 * \infty$

The limit $\lim_{x \to c} f(x)g(x)$ is a $0 * \infty$ **indeterminate form** if both

- $\lim_{x \to c} f(x) = 0$ and
- $\lim_{x \to c} g(x) = \infty$

Idea: Rewrite the product as a quotient so we can use L'Hospital's Rule

Example 6.

Evaluate

$$\lim_{x \to 0^+} x \ln x$$

Indeterminate Differences

Indeterminate Form $\infty - \infty$

The limit $\lim_{x \to c} [f(x) - g(x)]$ is a $\infty - \infty$ **indeterminate form** if both

- $\lim_{x \to c} f(x) = \infty$ and
- $\lim_{x \to c} g(x) = \infty$

Idea: Rewrite the difference as a quotient so we can use L'Hospital's Rule or rewrite as a product.

Example 7.

Evaluate

$$\lim_{x \to \pi/2^-} (\sec x - \tan x)$$

Indeterminate Powers

Indeterminate Forms 0^0 ∞^0 1^∞

The limit $\lim_{x \to c} f(x)^{g(x)}$ is an **indeterminate form** if both

- $\lim_{x \to c} f(x) = 0$ and $\lim_{x \to c} g(x) = 0$
- $\lim_{x \to c} f(x) = \infty$ and $\lim_{x \to c} g(x) = 0$
- $\lim_{x \to c} f(x) = 1$ and $\lim_{x \to c} g(x) = \infty$

Idea: Use the natural logarithm to convert the power into a product.

Example 8.

Evaluate

$$\lim_{x \to 0^+} (1 + \sin 4x)^{\cot x}$$

Example 9.

Evaluate

$$\lim_{x \to 0^+} x^x$$