Limit Laws

The overlying strategy developed here is to use the limit laws to convert an unknown limit into limits we can compute.

Limit Laws

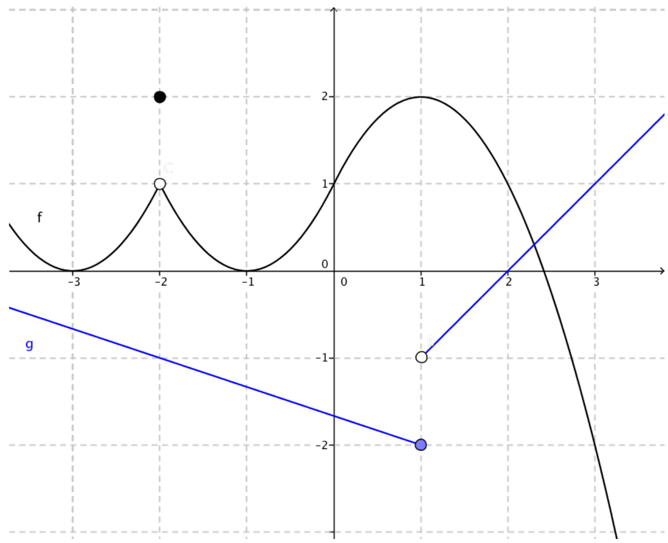
Suppose that c is a constant and the limits $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist. Then

- **(Sum Law)**
- **(Difference Law)**
- **(Constant Multiple Law)**
- **(Product Law)**
- **(Quotient Law)**

Example 1

Use the Limit Laws and the graphs of f and g to evaluate the following the limits, if they exist.

$$\lim_{x\to -2}[f(x)+5g(x)] \qquad \lim_{x\to 1}[f(x)g(x)] \qquad \lim_{x\to 2}\left[\frac{f(x)}{g(x)}\right]$$



Two Special Limits

- **(Constant Function)**

- **(Identity Function)**

Power and Root Laws

- **(Power Law)** Where n is a positive integer
  - (Special Case $f(x) = x$)

- **(Root Law)** Where n is a positive integer
  - (Special Case $f(x) = x$)

Example 2

Evaluate the following limit and justify each step.

$$\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x}$$

Direct Substitution Property

If f is a polynomial or rational function and a is in the domain of f, then

$$\lim_{x \to a} f(x) = f(a)$$

Limits Involving Quotients (Introduction)

One of the recurring problems in this course is finding the limit of the quotient of two functions.

$$\textbf{(Quotient Law)} \qquad \lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \qquad \text{if } \lim_{x\to a} g(x) \neq 0$$

- $\lim_{x\to a} g(x) \neq 0$

- $\lim_{x\to a} g(x) = 0$

We need a limit law that allows us to perform algebraic operations without changing the limit.

Proposition

If $f(x) = g(x)$ when $x \neq a$, then $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$, provided the limits exist.

Example 3 (Factor and Cancel)

Find $\lim_{x\to 1} \dfrac{x^2-1}{x-1}$.

Example 4 (Expand and Cancel)

Find $\lim_{h \to 0} \dfrac{(3+h)^2 - 9}{h}$.

Example 5 (Multiply by Conjugate)

Find $\lim_{t \to 0} \dfrac{\sqrt{t^2+9} - 3}{t^2}$.

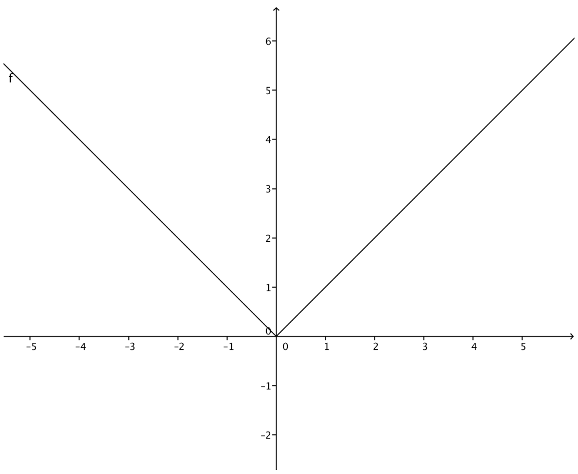
Limit Laws and One-Sided Limits

Proposition

All the Limit Laws also hold for both the right and left hand limits.

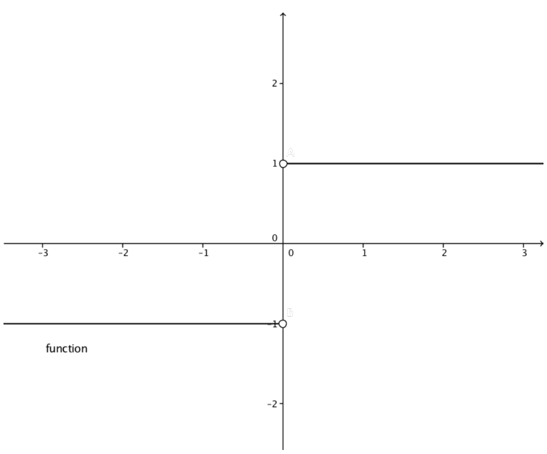
Example 6 (Piecewise Function)

Determine whether $\lim_{x \to 0} |x|$ exists.



Example 7 (Piecewise Function)

Determine whether $\lim_{x \to 0} \frac{|x|}{x}$ exists.



Example 8 (Piecewise Function)

For the following function determine whether $\lim_{x \to 4} f(x)$ exists:

$$f(x) = \begin{cases} \sqrt{x-4} & \text{if } x > 4 \\ 8 - 2x & \text{if } x < 4 \end{cases}$$

Comparison Properties of Limits

Theorem

If $f(x) \leq g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a, then

$$\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x).$$

Squeeze Theorem

If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$, then

$$\lim_{x \to a} g(x) = L.$$

Example 9 (Squeeze Theorem)

Show that $\lim_{x\to 0} x^2 \sin\dfrac{1}{x} = 0$.