The Second Derivative and the Shape of the Graph

In this section our goal is to deduce facts about a function from information about its second derivative.

The Shape of a Graph

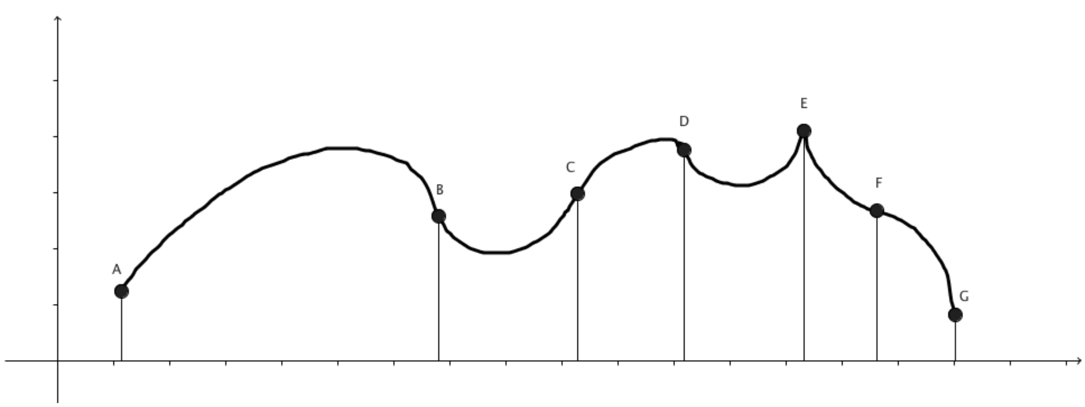
Concavity

On the interval (a, b) the graph of a function f is

- **concave upward** if the graph of f lies above all of its tangents on (a, b) and
- **concave downward** if the graph of f lies below all of its tangents on (a, b).

Inflection Point

- An **inflection point** of a function f is a point $P = (d, f(d))$ on the graph of f where either
  - the graph switches from concave upward to concave downward or
  - the graph switches from concave downward to concave upward.
- An **inflection number** of a function f is the x coordinate of the inflection point.



Finding Intervals of Concavity

As the next theorem suggests, finding the intervals of concavity of a function f is equivalent to finding the intervals on which its second derivative f'' is positive or negative.

Concavity Test

For the interval (a, b), if $f''(x) > 0$, then f is concave upward, and if $f''(x) < 0$, then f is concave downward.

$f''(x)$	$f'(x)$	$f(x)$	Examples

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Partition Numbers of f''

A **partition number** of the function f'' is a real number c such that either

1. $f''(c) = 0$ or
2. $f''(c)$ DNE

Example 1.

Find the intervals of concavity and any inflection points for the function $f(x) = x^4 - 4x^3$.

Example 2.

Find the intervals of concavity and any inflection points for the function $f(x) = (1-x)^{1/3}$.

Example 3.

Find the intervals of concavity and any inflection points for the function

$$f(x) = \frac{1}{(x-2)}$$

Finding Local Maximum and Minimum Values

Strategy:

Second Derivative Test for Local Extrema

Let c be a critical number of $f(x)$ such that $f'(c) = 0$.

- If $f''(c) > 0$ then $f(c)$ is a local minimum.
- If $f''(c) < 0$ then $f(c)$ is a local maximum.

Example 4.

Find the local minimum and maximum values of the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$.

Example 5.

Find the local minimum and maximum values of the function $f(x) = x^4 - 4x^3$.

Finding Absolute Maximum and Minimum Values

- We have already seen how to use the first derivative and the **Closed Interval Method** to find the absolute extreme values for a function defined on a closed interval.

- We have also seen that under special circumstances the first derivative can be used to find an absolute extreme value for a function defined on an open interval. There is a similar result for the second derivative.

Second Derivative Test for Absolute Extreme Values

Let c be the **only** critical number of a continuous function f and $f'(c) = 0$.

- If $f''(c) < 0$ then $f(c)$ is the absolute maximum value of f.
- If $f''(c) > 0$ then $f(c)$ is the absolute minimum value of f.

Example 6.

Find any absolute extreme values of the function

$$A(x) = 2400 - 2x^2$$