The Chain Rule

In this section we determine how to calculate the derivative of the composition of two functions in terms of the original functions and their derivatives.

Introduction

Composite Functions

Composite Functions

A function F is a **composite** of functions f and g if

$$F(x) = f\left(g(x)\right)$$

The domain of F is the set of all numbers x such that x is in the domain of g and $g(x)$ is in the domain of f.

Example 1.

Let $f(x) = \sqrt{x}$ and $g(x) = \sqrt{2-x}$. Find $f(g(x))$ and $g(f(x))$ and their domains.

Example 2.

Write each function as a composite of two simpler functions.

1. $y = 100e^{0.04x}$

2. $y = \sqrt{x^2+3}$

The Chain Rule

Chain Rule

If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is:

1. differentiable at x and
2. F' is given by the product

$$F'(x) = f'(g(x))g'(x)$$

Example 3.

Find $F'(x)$ if $F(x) = \sqrt{x^2 + 3}$

Example 4.

Differentiate:

1. $y = \sin(x^2)$

2. $y = \sin^2 x$

Special Cases

General Power Rule

If $u(x)$ is a differentiable function, n is any real number, and

$$y = f(x) = (u(x))^n$$

then

$$f'(x) = n\,(u(x))^{n-1} u'(x)$$

General Exponential Rule

If $u(x)$ is a differentiable function, n is any real number, and

$$y = f(x) = e^{u(x)}$$

then

$$f'(x) = e^{u(x)} u'(x)$$

General Trigonometric Rule

If $u(x)$ is a differentiable function, n is any real number, and

$$y = f(x) = \sin(u(x))$$

then

$$f'(x) = \cos(u(x))\,u'(x)$$

Example 5.

Differentiate $y = (x^3 - 1)^5$.

Example 6.

Differentiate $f(x) = \dfrac{1}{\sqrt[3]{x^2 + x + 1}}$.

Example 7.

Differentiate $g(t) = \left(\dfrac{t - 2}{2t + 1} \right)^9$.

Example 8.

Differentiate $y = (2x+1)^5(x^3 - x + 1)^4$.

Example 9.

Differentiate $y = e^{\sin x}$.

Example 10.

Differentiate $y = a^x$.

Using Chain Rule More Than Once

Example 11.

Differentiate $f(x) = \sin(\cos(\tan x))$

Example 12.

Differentiate $y = e^{\sec 3\theta}$