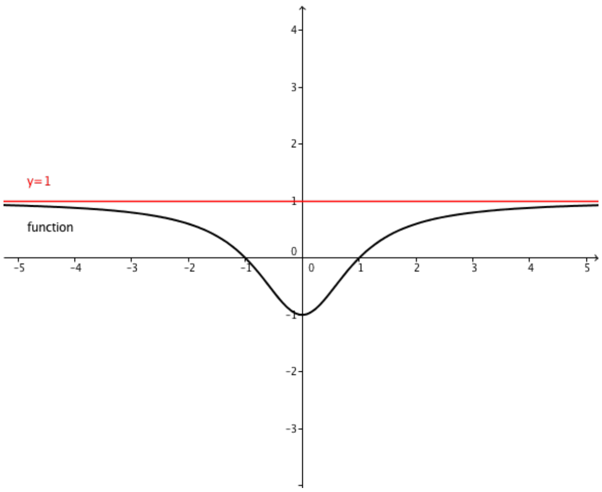
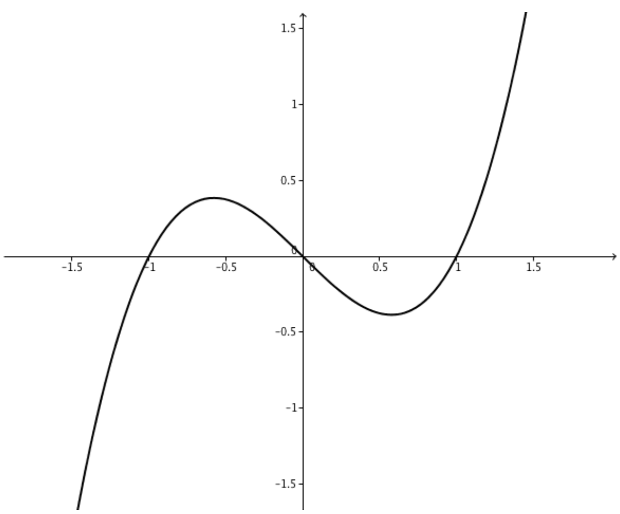
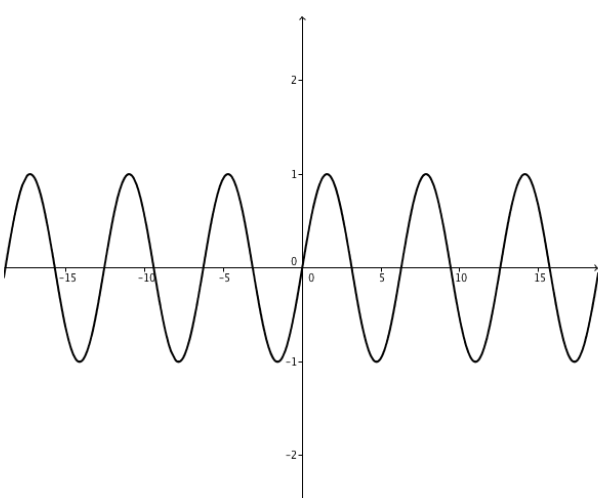
Limits at Infinity

Introduction

So far we have considered limits as $x \to a$ for some number a, now we consider the long-term behavior of a function as x gets very large positive and very large negative.







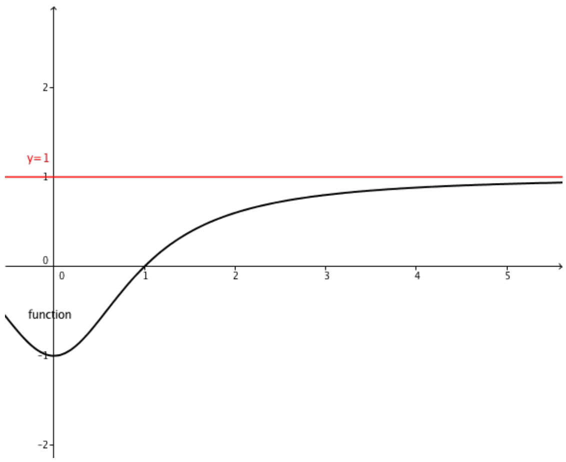
Horizontal Asymptotes

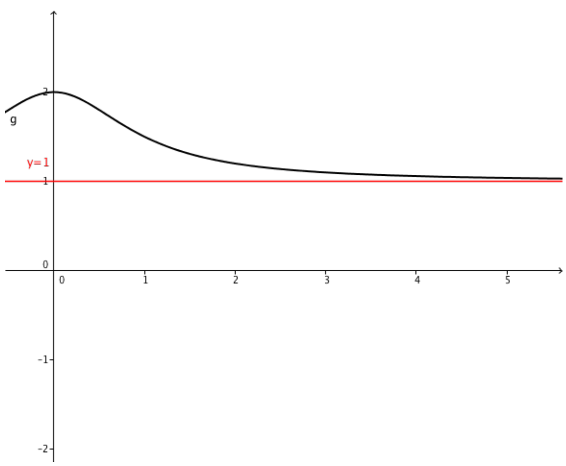
Limit at Infinity

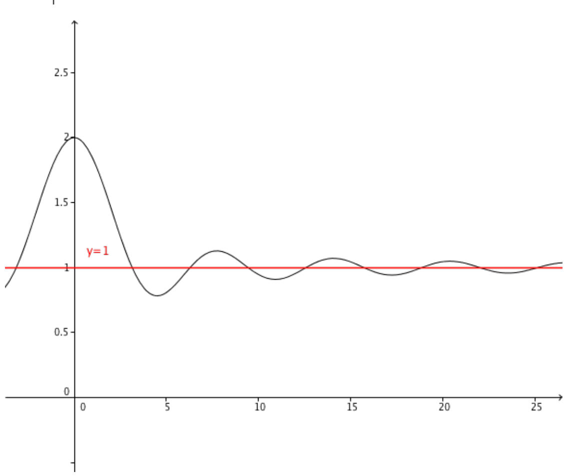
Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

means the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large.





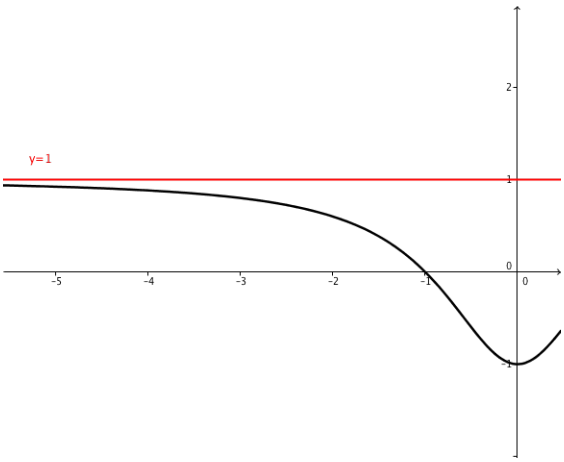


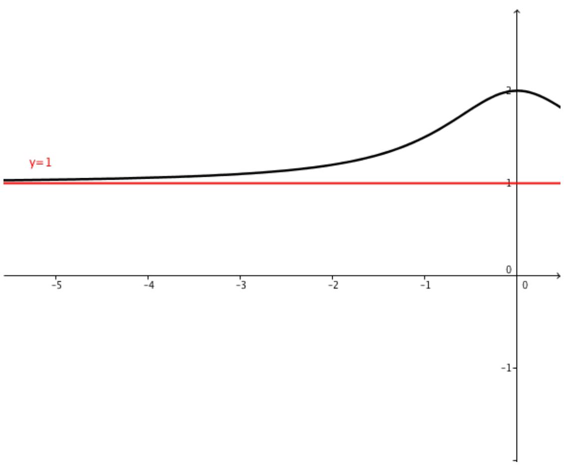
Limit at Negative Infinity

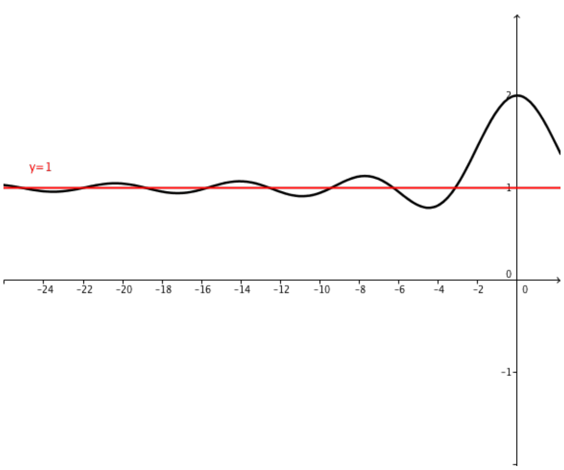
Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x \to -\infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large negative.







Horizontal Asymptotes

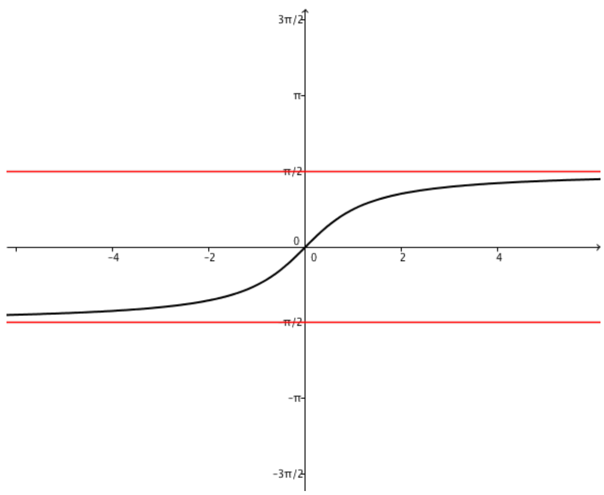
The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = L \qquad \text{or} \qquad \lim_{x\to-\infty} f(x) = L$$

Example 1.

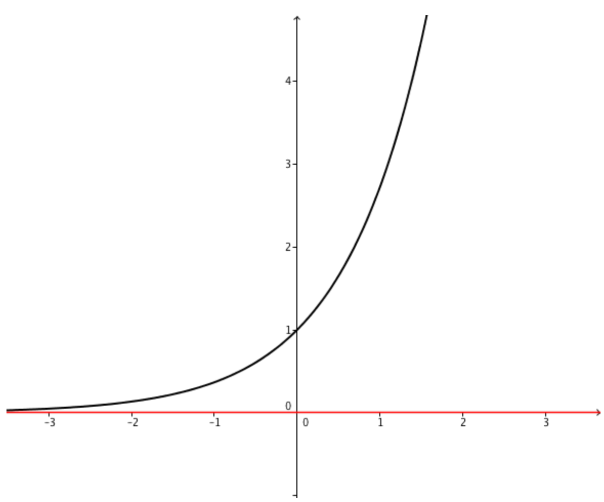
- The function $\arctan x$ has two horizontal asymptotes since:

$$\lim_{x\to-\infty} \arctan x = -\frac{\pi}{2} \qquad\qquad \lim_{x\to\infty} \arctan x = \frac{\pi}{2}$$



- The function e^x has one horizontal asymptote since:

$$\lim_{x\to-\infty} e^x = 0$$



Example 2.

Evaluate the following limits:

- $\lim_{x \to \infty} \frac{1}{x}$

- $\lim_{x \to -\infty} \frac{1}{x}$

Limit Laws

All of the Limit Laws are also valid for limits at infinity except:

Theorem

- If $r > 0$ is a rational number, then

$$\lim_{x \to \infty} \frac{1}{x^r} = 0$$

- If $r > 0$ is a rational number such that x^r is defined for all x, then

$$\lim_{x \to -\infty} \frac{1}{x^r} = 0$$

Example 3.

Evaluate

$$\lim_{x \to \infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1}$$

Strategy to Find All Asymptotes

- Horizontal Asymptotes
  - Compute $\lim_{x \to \infty} f(x)$
  - Compute $\lim_{x \to -\infty} f(x)$
- Vertical Asymptotes
  - Find all values of x that give a zero in the denominator
  - Compute both left and right hand limits of f as x approaches each zero.

Example 4.

Find the horizontal and vertical asymptotes of the graph of the function

$$f(x) = \frac{\sqrt{2x^2 + 1}}{3x - 5}$$

Converting Limits to Limits at Infinity

Example 5.

Evaluate $\lim_{x\to 2^+} \arctan\left(\frac{1}{x-2}\right)$.

Strategy for converting limits

1. Let $t =$inside function **(change of variables)**
2. Compute $\lim_{x\to a} t$.
3. Compute the limit of the function with the changed variables.

Example 6.

Evaluate $\lim_{x\to 0^-} e^{1/x}$

Infinite Limits at Infinity

Infinite Limit at Infinity

Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x\to\infty} f(x) = \infty$$

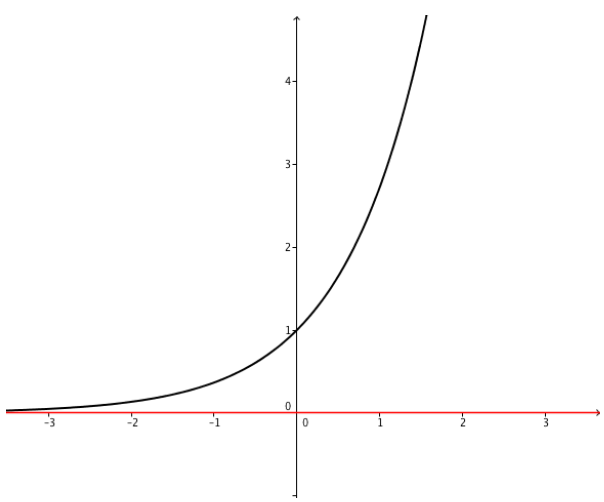
means the values of $f(x)$ can be made arbitrarily large positive by taking x sufficiently large.

Example 7.

Find $\lim_{x\to\infty} x^3$ and $\lim_{x\to-\infty} x^3$.

- The function e^x has one infinite limit at infinity since:

$$\lim_{x\to\infty} e^x = \infty$$



Overview of Infinite Limit Techniques

Example 8. **($1/x^n$ Method)**

Find $\lim_{x\to\infty} \frac{x^2+x}{3-x}$.

Example 9. **(Conjugate Method)**

Compute $\lim_{x\to\infty} \left(\sqrt{x^2+1}-x\right)$.

Example 10. **(Factor Method)**

Compute $\lim_{x\to\infty} \left(x^2-x\right)$.