The Intuitive Definition of a Limit

Example 1.

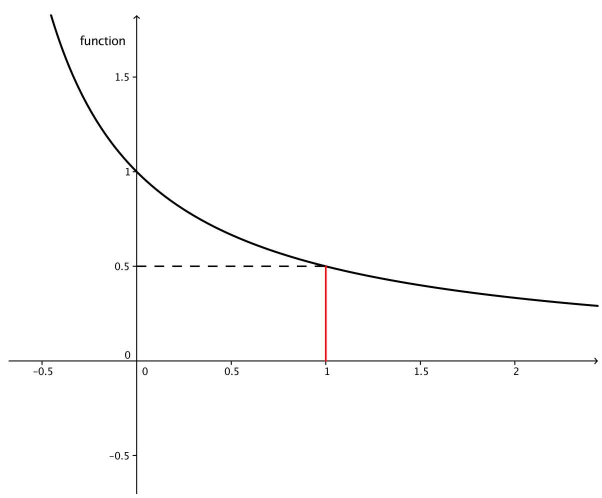
Investigate the behavior of the function $f(x) = \dfrac{1}{x+1}$ when x is near 1.

Numerical Method: Calculate values of $f(x)$ using: x terms that are slightly larger than 1 and x terms that are slightly smaller than 1.

$x < 1$	$f(x)$
0.5	0.666667
0.9	0.526316
0.99	0.502513
0.999	0.500250
0.9999	0.500025

$x > 1$	$f(x)$
1.5	0.400000
1.1	0.476190
1.01	0.497512
1.001	0.499750
1.0001	0.499975

Graphical Method: Interpret the graph of $f(x)$



Limit of a Function (Intuitive Definition)

Suppose $f(x)$ is defined when x is near a.

If we can make the values of $f(x)$ as close to L as we want
by taking x sufficiently close to a but not equal to a

Then we write

$$\lim_{x \to a} f(x) = L$$

Example 2.

Guess the value of

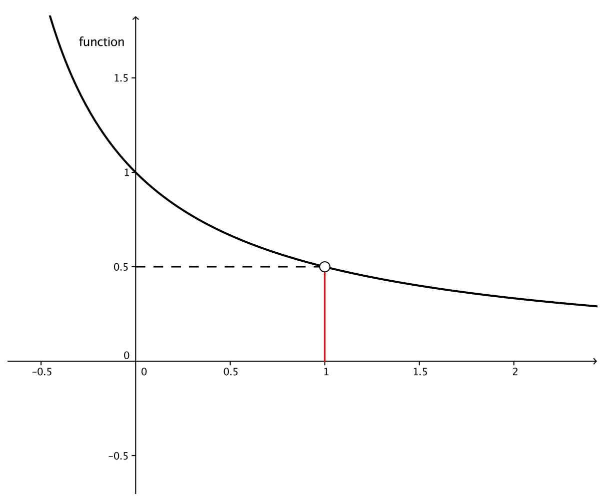
$$\lim_{x \to 1} \frac{x-1}{x^2-1}$$

Numerical Method: Calculate values of $f(x)$ as x gets closer to a from the left and right.

$x < 1$	$f(x)$
0.5	0.666667
0.9	0.526316
0.99	0.502513
0.999	0.500250
0.9999	0.500025

$x > 1$	$f(x)$
1.5	0.400000
1.1	0.476190
1.01	0.497512
1.001	0.499750
1.0001	0.499975

Graphical Method: Interpret the graph of $f(x)$



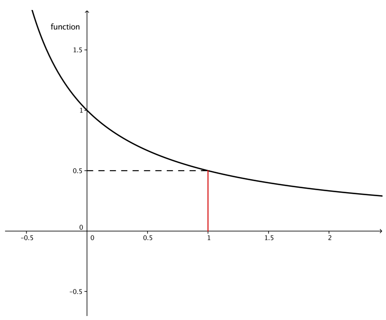
Example 3.

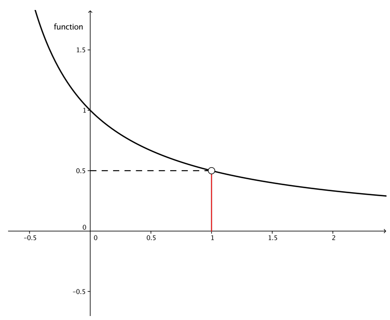
Guess the value of

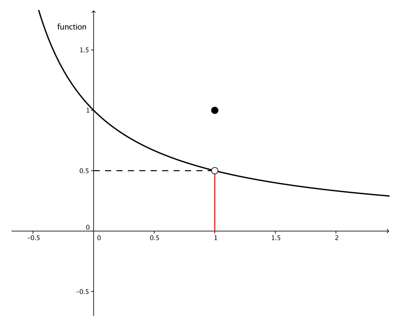
$$\lim_{x\to 1} f(x)$$

where

$$f(x) = \frac{1}{x+1} \qquad f(x) = \frac{x-1}{x^2-1} \qquad f(x) = \begin{cases} \frac{x-1}{x^2-1} & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases}$$







Example 4.

Guess the value of

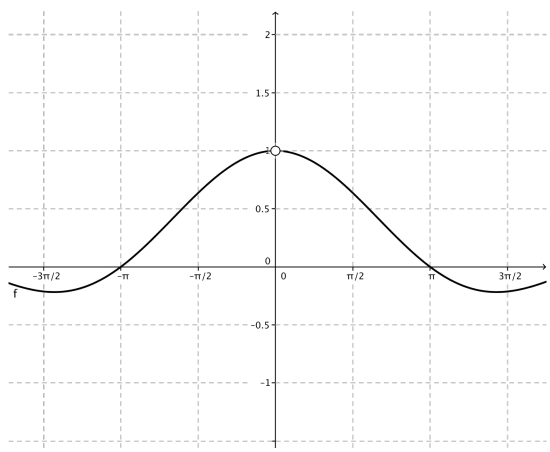
$$\lim_{x \to 0} \frac{\sin x}{x}$$

Numerical Method: Calculate values of $f(x)$ as x gets closer to a from the left and right.

$x < 0$	$f(x)$
-0.5	0.95885108
-0.1	0.99833417
-0.05	0.99958339
-0.01	0.99998333
-0.001	0.99999983

$x > 0$	$f(x)$
0.5	0.95885108
0.1	0.99833417
0.05	0.99958339
0.01	0.99998333
0.001	0.99999983

Graphical Method: Interpret the graph of $f(x)$



Caution

The methods used in these examples are for intuitive purposes. Even though they usually work, there are cases where they might lead to an incorrect conclusion. Some issues that you might encounter:

- Rounding Error: by design calculators automatically round very small numbers. As an example try using your calculator to evaluate $-3 + \sqrt{(0.00000001)^2 + 9}$
- Incorrect Guess: if the limit is not an integer it is usually difficult to determine the value using a graph or table of values.

One-Sided Limits

Left Hand Limits

Left-hand limit of $f(x)$ as x approaches a (from the left)

$$\lim_{x\to a^-} f(x) = L$$

We can make the values of $f(x)$ arbitrarily close to L by using only $x < a$.

Right Hand Limits

Right-hand limit of $f(x)$ as x approaches a (from the right)

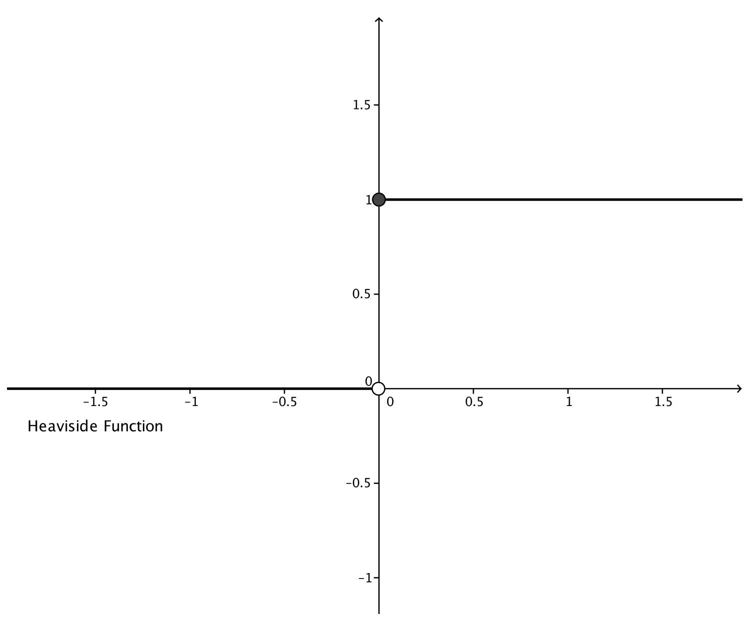
$$\lim_{x\to a^+} f(x) = L$$

We can make the values of $f(x)$ arbitrarily close to L by using only $x > a$.

Example 5.

Investigate the behavior of the Heaviside function H near $x = 0$

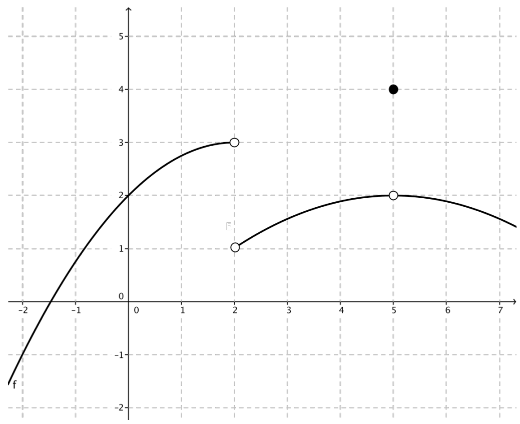
$$H(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$



Theorem

$$\lim_{x\to a} f(x) = L \quad \text{if and only if} \quad \lim_{x\to a^-} f(x) = L \quad \text{and} \quad \lim_{x\to a^+} f(x) = L$$

Example 6.



Find the values of the following:

$\lim_{x\to 2^-} g(x)$	$\lim_{x\to 2^+} g(x)$	$\lim_{x\to 2} g(x)$	$g(2)$
$\lim_{x\to 5^-} g(x)$	$\lim_{x\to 5^+} g(x)$	$\lim_{x\to 5} g(x)$	$g(5)$

How Can a Limit Fail to Exist?

There are many different ways that the limit can fail to exist for a particular a value: