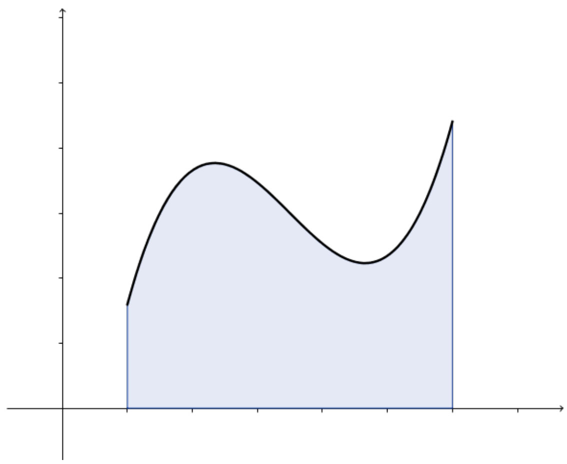
The Intuitive Definition of the Definite Integral

Definite Integral: Positive Function

Let f be a continuous function such that $f(x) \geq 0$ on the interval $[a, b]$.

We define the **definite integral of f from a to b** as the **area** of the region between the graph of f and the x-axis:

$$\text{Area} = \int_a^b f(x)\,dx$$

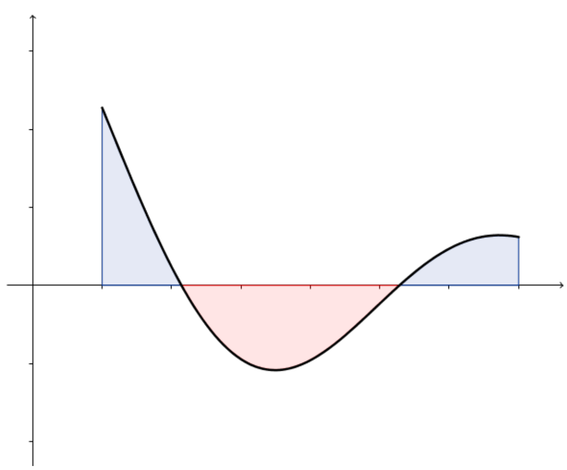


Definite Integral: General Function

Let f be a continuous function on the interval $[a, b]$.

We define the **definite integral of f from a to b** as the **net area** of the region between the graph of f and the x-axis:

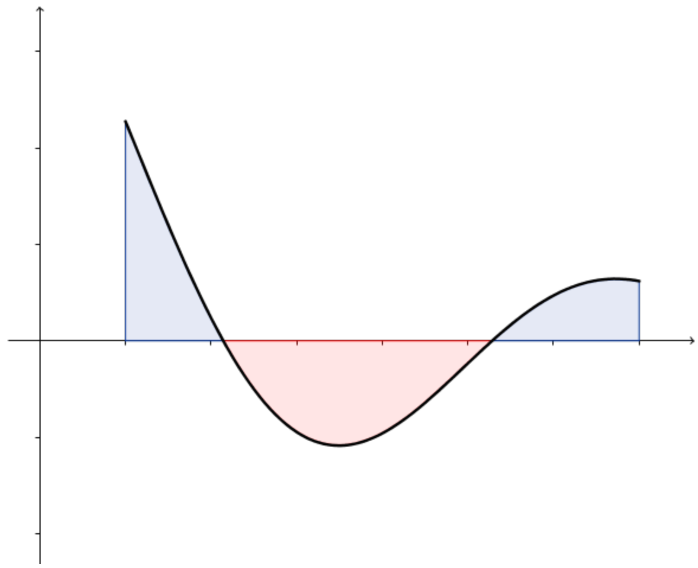
$$\text{Net Area} = \int_a^b f(x)\,dx$$



Interpretation of the Definite Integral

Example 1.

Calculate the definite integrals by referring to the figure:



1. $\displaystyle\int_a^b f(x)\,dx$

2. $\displaystyle\int_a^c f(x)\,dx$

3. $\displaystyle\int_b^c f(x)\,dx$

Example 2.

Evaluate the following integrals by interpreting each in terms of areas.

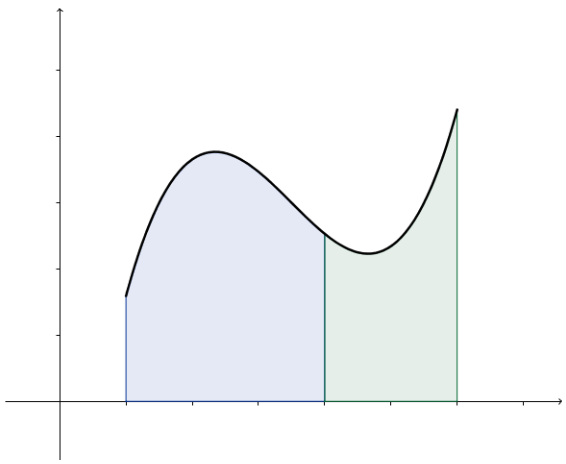
1. $\int_0^1 \sqrt{1-x^2}\, dx$

2. $\int_0^3 (x-1)\, dx$

Properties of the Definite Integral

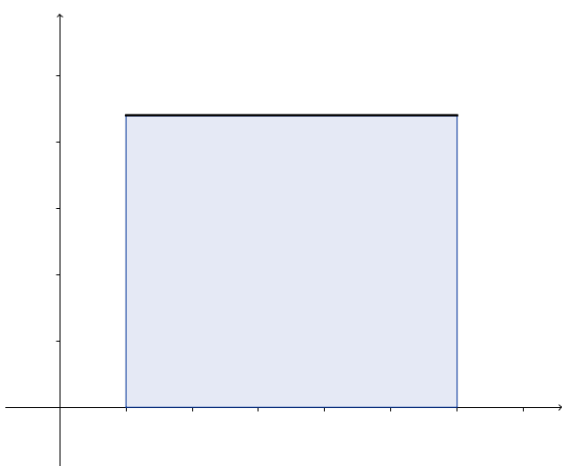
Properties of Definite Integrals: Limits of Integration

- $\int_a^a f(x)\,dx = 0$
- $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
- $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$



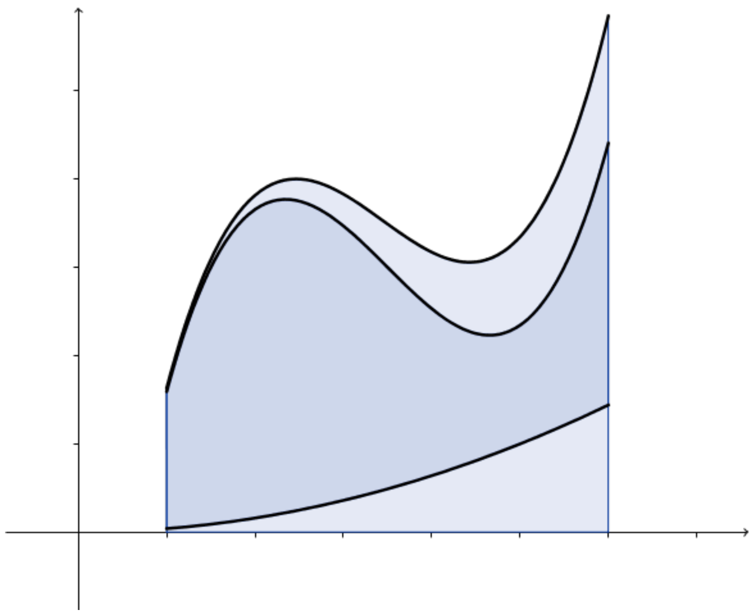
Properties of Definite Integrals: Constant Function

- $\int_a^b c\,dx = c \cdot (b - a)$ $\quad$ c is any constant



Properties of Definite Integrals: Integrand

- $\int_a^b k \cdot f(x)\, dx = k \cdot \int_a^b f(x)\, dx$
- $\int_a^b \left[f(x) + g(x)\right]\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx$
- $\int_a^b \left[f(x) - g(x)\right]\, dx = \int_a^b f(x)\, dx - \int_a^b g(x)\, dx$



Example 3. [Using Properties of the Definite Integral]

Use the fact that

$$\int_0^2 x\,dx = 2 \qquad \int_0^2 x^2\,dx = \frac{8}{3} \qquad \int_2^3 x^2\,dx = \frac{19}{3}$$

to determine the following integrals:

1. $\int_0^2 12x^2\,dx$

2. $\int_0^2 (2x - 6x^2)\,dx$

3. $\int_3^2 x^2\,dx$

4. $\int_5^5 3x^2\,dx$

5. $\int_0^3 3x^2\,dx$

Example 4. [Using Properties of the Definite Integral]

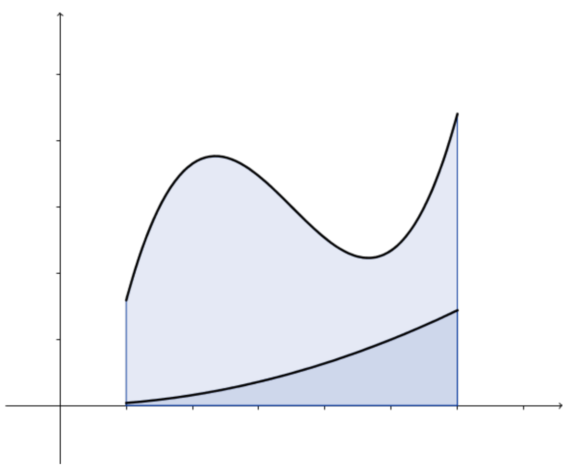
Use the fact that for some function $f(x)$: $\int_0^{10} f(x)\,dx = 17$ $\int_0^8 f(x)\,dx = 12$

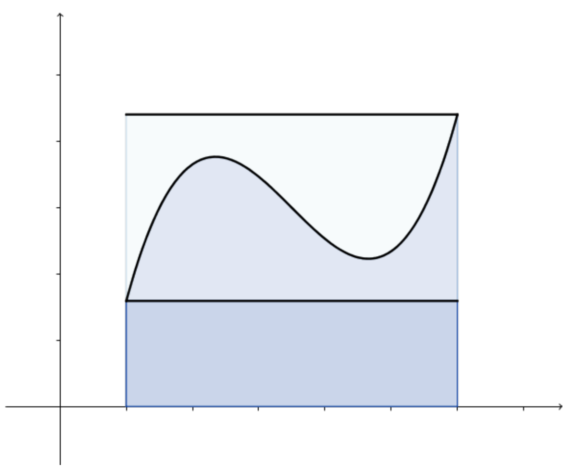
to determine the integral $\int_8^{10} f(x)\,dx$

Comparison Properties of the Integral

Properties of Definite Integrals: Comparisons

- If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq 0$
- If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$
- If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$





Example 5. [Using the Comparison Properties]

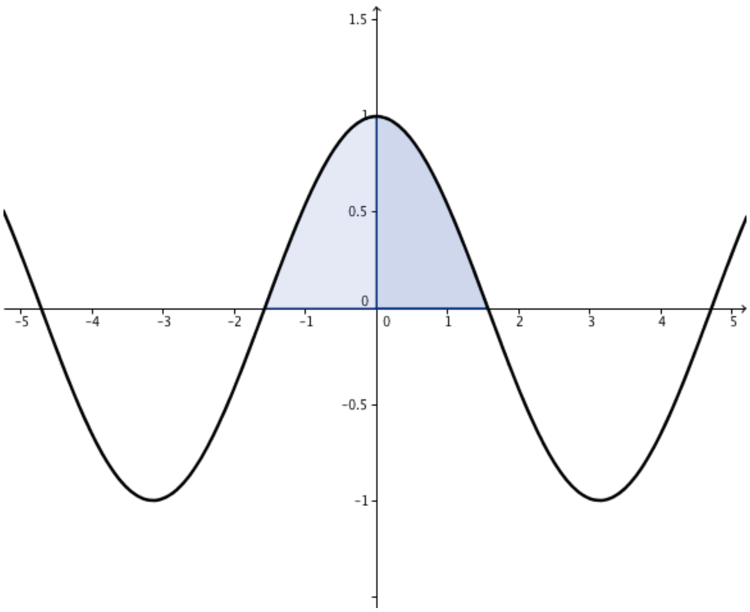
Use the comparison properties to estimate $\int_0^1 e^{-x^2}\,dx$

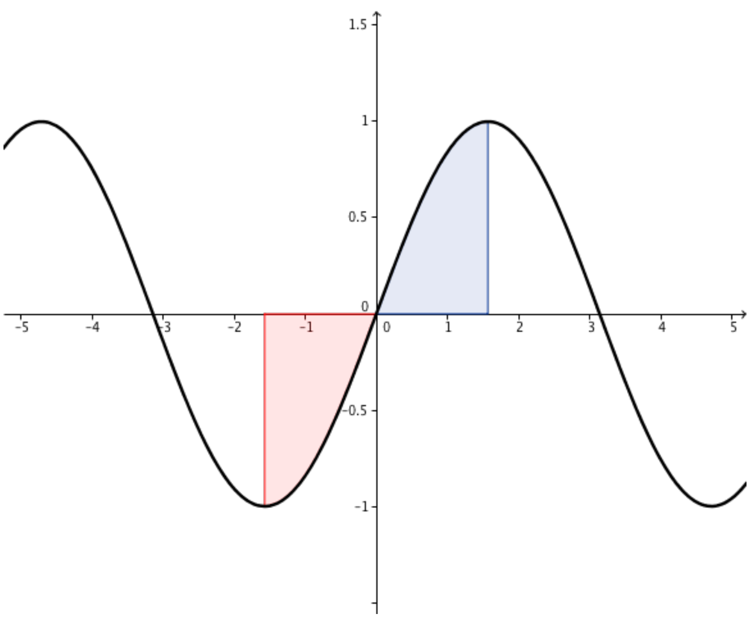
Symmetry and the Definite Integral

Definite Integrals of Symmetric Functions

Suppose f is continuous on $[-a, a]$.

- If f is even $[f(-x) = f(x)]$, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$
- If f is odd $[f(-x) = -f(x)]$, then $\int_{-a}^{a} f(x)\,dx = 0$





Example 6. [Definite Integrals and Symmetry]

1. Use the fact that $\int_0^2 (x^2+1)\,dx = \frac{14}{3}$ to evaluate the integral $\int_{-2}^2 (x^2+1)\,dx$

2. Evaluate $\int_{-1}^1 \frac{\tan x}{1+x^2+x^4}\,dx$