The Precise Definition of a Limit

How Can We Measure the Closeness of Two Numbers?

Recall the intuitive definition of the limit of a function....

Limit of a Function (Intuitive Definition)

Suppose $f(x)$ is defined when x is near a.

If we can make the values of $f(x)$ as close to L as we want
by taking x sufficiently close to a but not equal to a

Then we write

$$\lim_{x \to a} f(x) = L$$

Example 1.

Using the intuitive definition of a limit we conclude that

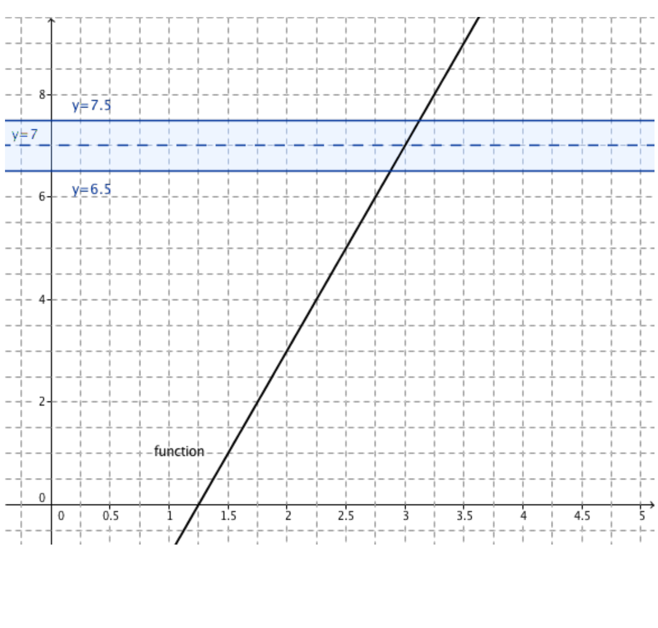
$$\lim_{x\to 3}(4x-5)=7$$

1. How close to 3 does x have to be so that $4x-5$ differs from 7 by less than 0.5?

2. How close to 3 does x have to be so that $4x-5$ differs from 7 by less than 0.1?

3. Use a graph to find a number δ such that

$$\text{if} \quad 0<|x-3|<\delta \qquad \text{then} \qquad |(4x-5)-7|<0.5$$



NOTE: For the following definition we will assume that f is a function defined on some open interval that contains the number a, except possibly at a itself.

Limits

Definition of Limit

The **limit of** $f(x)$ **as** x **approaches** a **is** L and we write

$$\lim_{x \to a} f(x) = L$$

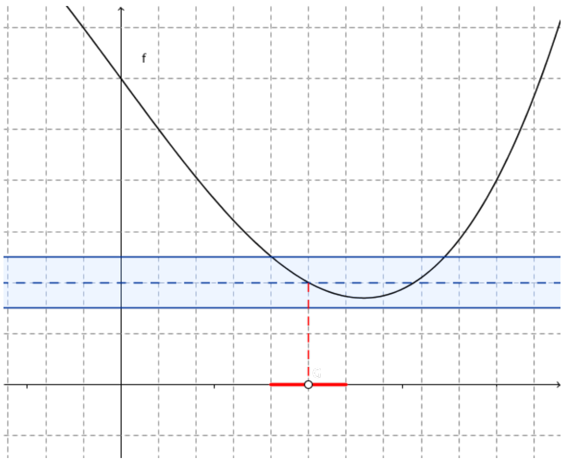
If for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

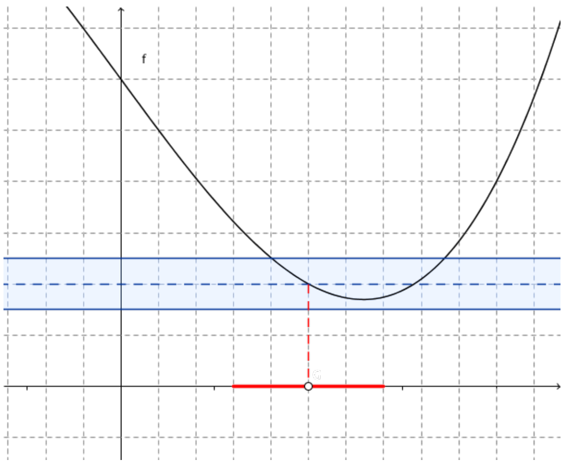
$$\text{if} \quad 0 < |x - a| < \delta \qquad \text{then} \qquad |f(x) - L| < \varepsilon$$

Comparison with the Intuitive Definition:

Arrow Diagram Representation:

Graph Representation:

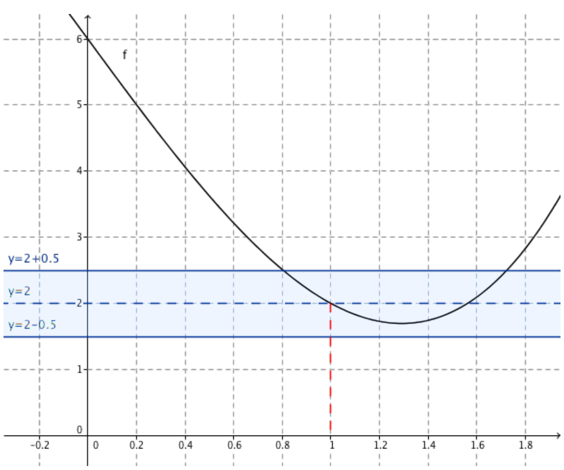




Example 2.

Use a graph to find a number δ such that

$$\text{if} \quad 0 < |x-1| < \delta \qquad \text{then} \qquad \left|(x^3 - 5x + 6) - 2\right| < 0.5$$



The last example is meant to show us that the limit of the function is most likely 2 but it does not ***prove*** that this is the limit. This argument needs to work for any ε not just 0.5. The next example shows how an actual mathematical proof proceeds.

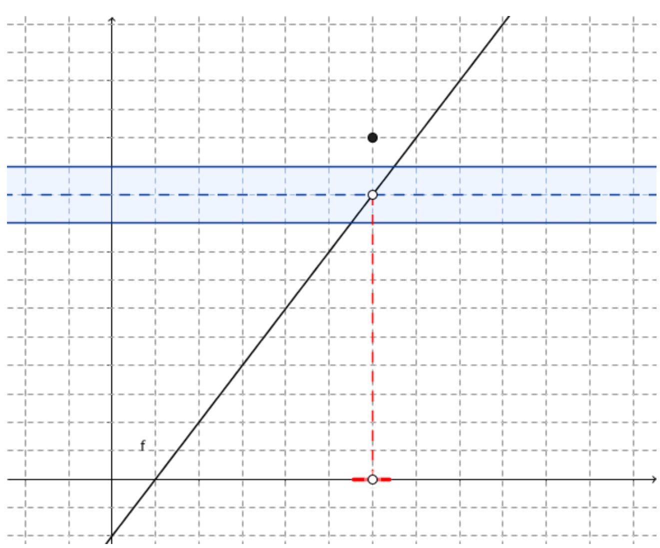
Example 3.

Prove that $\lim_{x \to 3} (4x - 5) = 7$.

Example 4.

Prove that $\lim_{x \to 3} f(x) = 5$ where

$$f(x) = \begin{cases} 2x - 1 & \text{if} \quad x \neq 3 \\ 6 & \text{if} \quad x = 3 \end{cases}$$



Usually proving the limit of a function directly is difficult which is why the Limit Laws can be very helpful. We can prove all of the Limit Laws using the limit definition. Recall the Sum Law:

Sum Law

If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$ then

$$\lim_{x \to a} [f(x) + g(x)] = L + M$$

One-Sided Limits

Definition of Left-Hand Limit

$$\lim_{x \to a^-} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \quad a - \delta < x < a \qquad \text{then} \qquad |f(x) - L| < \varepsilon$$

Definition of Right-Hand Limit

$$\lim_{x \to a^+} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \quad a < x < a + \delta \qquad \text{then} \qquad |f(x) - L| < \varepsilon$$

Example 5.

Prove that $\lim_{x \to 0^+} \sqrt{x} = 0$.