ringland[01] ex2

MTH 418/518 PDE Exam 2 Spring 2017

* You are allowed a simple scientific *non-graphing, non-symbolic* calculator.

* You may NOT use a cellphone, tablet, computer, or any other communication device.

* The time allowed for this exam is 80 minutes (unless special arrangements have been made with the instructor).

Student UBIT name:

ringland

Answer the questions in the spaces provided. Do not write on the back of any page. Do not write on the QR code printed at the top right corner of each sheet. If you need extra space, use the blank pages at the back of the booklet. Write legibly and express yourself clearly to get the maximum 4 points for "style". This exam has 4 questions for undergrads and 4 questions for grad students for a total of 50+4 points.

Solutions

1. (a) (4 points) Define the Fourier series with respect to the interval $[-L, L]$ of an integrable function f.

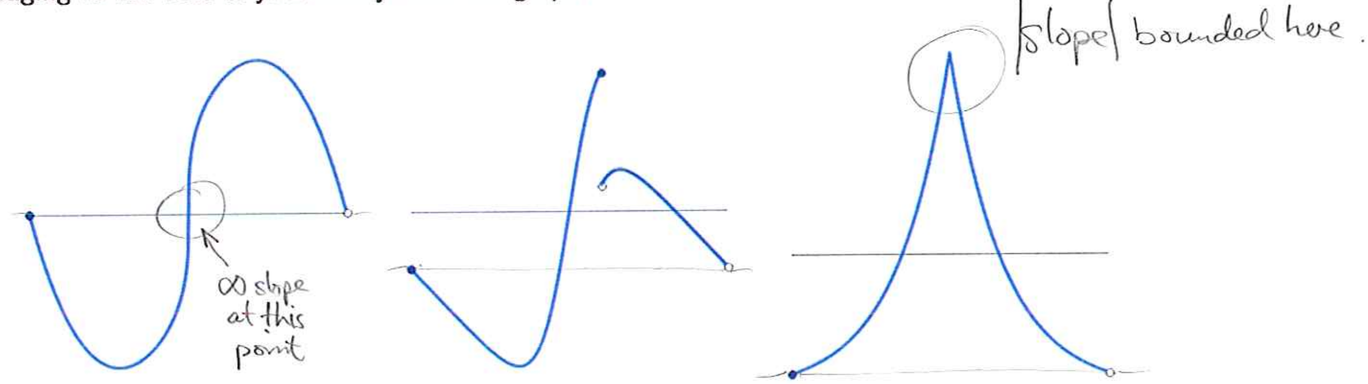
$$FS[f](x) = \sum_{n=0}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

Where

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx, \quad a_{n\geq 1} = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx, \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

(b) (6 points) Reminders: A function f is is said to be *piecewise continuous* on an interval $[a, b]$ if the interval can be partitioned by a finite number of points $a = x_0 < x_1 < \ldots < x_n = b$ so that (i) f is continuous on each subinterval (x_i, x_{i+1}), and (ii) f has a finite one-sided limits at each x_i. A function f is said to be *piecewise smooth* if f is piecewise continuous and so is f'.

Consider the 3 functions graphed below, and answer the questions below them, judging to the best of your ability from the graphs.



Circle yes or no.

continuous?	yes/no	yes/no	yes/no
continuous derivative?	yes/no	yes/no	yes/no
continuous periodic extension?	yes/no	yes/no	yes/no
periodic extension has continuous derivative?	yes/no	yes/no	yes/no
piecewise continuous?	yes/no	yes/no	yes/no
piecewise smooth?	yes/no	yes/no	yes/no

(c) (2 points) If f is a piecewise smooth function, we have learned that its Fourier series f converges (to something finite) everywhere, and converges to f at points where f is continuous. To what does it converge at a point p where f is *not* continuous?

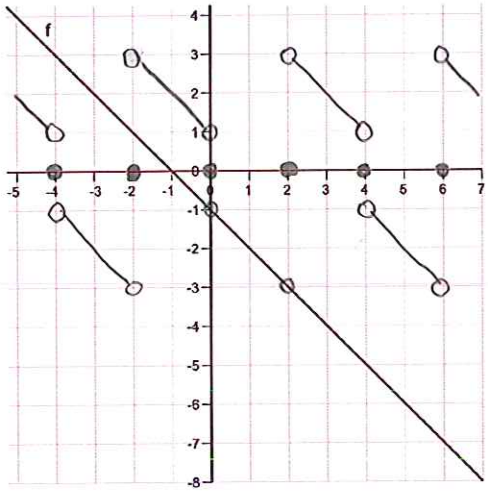
$$\frac{f(p_-) + f(p_+)}{2}$$

where

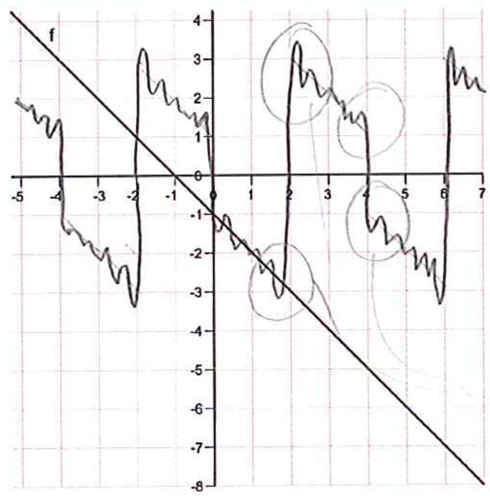
$$f(p_-) = \lim_{x \to p^-} f(x)$$

$$f(p_+) = \lim_{x \to p^+} f(x)$$

2. (a) (3 points) In the picture below, sketch the function to which the **Fourier sine series** with respect to the interval [0,2) of the graphed function f converges. Use open circles (∘) to denote non-included endpoints of intervals and filled circles (•) to denote included endpoints or isolated values.



(b) (3 points) In the picture below, sketch what you think the graph of the sum of the **first 20-or-so** terms of the Fourier sine series looks like.



Gibb's phenomenon
(overshoot)

(c) (10 points) Write down a formula for the function f used in (a) and (b), and compute the first 4 non-zero coefficients of the Fourier sine series with respect to the interval [0,2). (Hint to undergraduates: use integration by parts.)

$$f(x) = -1 - x$$

Sine series is $\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$

with $L = 2$, $b_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L}\, dx$

So

$$b_n = -\frac{2}{2}\int_0^2 (1+x)\sin\frac{n\pi x}{2}\, dx$$

Integrate by parts: $\int_a^b uv' = uv\Big|_a^b - \int u'v$

$$u(x) = 1+x, \quad v'(x) = \sin\frac{n\pi x}{2}$$

$$u'(x) = 1, \quad v(x) = -\frac{2}{n\pi}\cos\frac{n\pi x}{2}$$

So

$$b_n = -\left[(1+x)\left(\frac{-2}{n\pi}\right)\cos\frac{n\pi x}{2}\Big|_0^2 - \int_0^2 \left(\frac{-2}{n\pi}\right)\cos\frac{n\pi x}{2}\, dx\right]$$

$$= \frac{2}{n\pi}(1+x)\cos\frac{n\pi x}{2}\Big|_0^2 + \frac{2}{n\pi}\cdot\frac{2}{n\pi}\sin\frac{n\pi x}{2}\Big|_0^2$$

$$= \frac{2}{n\pi}\left[3\cos n\pi - 1\cdot 1\right] + 0$$

$$= \frac{2}{n\pi}\left[(-1)^n\cdot 3 - 1\right], \quad n = 1, 2, 3, \ldots$$

$$= \left\{\frac{2}{\pi}(-4),\ \frac{1}{\pi}(2),\ \frac{2}{3\pi}(-4),\ \frac{1}{2\pi}(2),\ \ldots\right\}$$

$$= \left\{\frac{-8}{\pi},\ \frac{2}{\pi},\ \frac{-8}{3\pi},\ \frac{1}{\pi},\ \ldots\right\}$$

3. Separation of variables in a wave-type PDE leads to solutions of the form $u(x,t) = h(t)\phi(x)$ where and $h''(t) + \lambda c^2 h(t) = 0$ and

$$\phi''(x) + \alpha\phi'(x) + \lambda\phi(x) = 0, \;\; \phi(0) = 0, \;\; \phi(L) = 0$$

Here α and c are positive constants, and λ is the separation constant.

(a) (4 points) Consider the eigenvalue problem for λ, ϕ. Multiply the DE for ϕ by $e^{\alpha x}$ and show that the resulting equation has Sturm-Liouville form $\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q\phi + \lambda\sigma\phi = 0$. Give formulas for each of p, q and σ.

$$\underbrace{e^{\alpha x}\phi'' + \alpha e^{\alpha x}\phi'}_{} + \lambda e^{\alpha x}\phi = 0$$

$$(e^{\alpha x}\phi')' \underbrace{- \alpha e^{\alpha x}\phi' + \alpha e^{\alpha x}\phi'}_{0} + \lambda e^{\alpha x}\phi = 0$$

$$(e^{\alpha x}\phi')' + \lambda e^{\alpha x}\phi = 0$$

Thus $p(x) = e^{\alpha x}$, $q(x) \equiv 0$, $\sigma(x) = e^{\alpha x}$

(b) (3 points) Use the Rayleigh quotient

$$RQ(\phi) = \left[\left(-p\phi\frac{d\phi}{dx}\right)\Big|_0^L + \int_0^L \left(p\left(\frac{d\phi}{dx}\right)^2 - q\phi^2\right)dx\right] \Big/ \int_0^L \sigma\phi^2 dx$$

to show the eigenvalues λ are strictly positive. Argue carefully that $\lambda = 0$ is not possible. Since $\phi(0) = 0 = \phi(L)$, boundary term is zero.

$$RQ[\phi] = \frac{\int_0^L e^{\alpha x}\phi'^2(x) - 0 \, dx}{\int_0^L e^{\alpha x}\phi^2(x)\, dx}$$

$$= \int_0^L e^{\alpha x}(\phi'(x))^2 \Big/ \int_0^L e^{\alpha x}(\phi(x))^2 dx.$$

Now $e^{\alpha x} > 0$ for all x.

$\phi'(x)^2 \geq 0$ for all x and $\phi' \not\equiv 0$ because otherwise $\phi = \text{const}$ and $\text{const} = 0$ from BCs hence $\phi \equiv 0$: trivial.

So $\int e^{\alpha x}(\phi'(x))^2 dx > 0$.

Likewise $\int (\phi(x))^2 dx > 0$.

So $RQ[\phi] > 0$.

Thus $\lambda_1 > 0$ and hence all $\lambda > 0$.

(c) (2 points) Explain why the result in (b) implies the product solutions oscillate sinusoidally in time.

Because $h'' + kh = 0$ with $k > 0$ has solutions $\cos\sqrt{k}t$ and $\sin\sqrt{k}t$.

(d) (6 points) Graduate students only.

Let $\alpha = 2\beta$ so now

$$\phi''(x) + 2\beta\phi'(x) + \lambda\phi(x) = 0, \quad \phi(0) = 0, \quad \phi(L) = 0.$$

Solve for the eigenvalues λ_n, $n = 1, 2, 3, \ldots$ assuming each $\lambda_n > \beta^2$.

Second order linear ODE with constant coeffs.

Char. eqn:

$$r^2 + 2\beta r + \lambda = 0$$

$$r = \frac{-2\beta \pm \sqrt{(2\beta)^2 - 4 \cdot 1 \cdot \lambda}}{2}$$

$$= \frac{-2\beta \pm \sqrt{4\beta^2 - 4\lambda}}{2} = -\beta \pm \sqrt{\beta^2 - \lambda}$$

Assuming $\lambda > \beta^2$,

$$r = -\beta \pm i\sqrt{\lambda - \beta^2}$$

General solution is

$$\phi(x) = c_1 e^{-\beta x} \cos\left(\sqrt{\lambda - \beta^2}\, x\right) + c_2 e^{-\beta x} \sin\left(\sqrt{\lambda - \beta^2}\, x\right)$$

Now $\phi(0) = 0$ and

$$\phi(0) = c_1 \underbrace{e^{-\beta 0}}_{1} \cdot 1 + \underbrace{c_2 e^{-\beta \cdot 0} \cdot 0}_{0}$$

So $c_1 = 0$ and

$$\phi(x) = c_2 e^{-\beta x} \sin\left(\sqrt{\lambda - \beta^2}\, x\right)$$

Also $\phi(L) = 0$ so

$$c_2 \underbrace{e^{-\beta L}}_{\text{nonzero}} \sin\left(\sqrt{\lambda - \beta^2}\, x\right) = 0,$$

hence need $\sqrt{\lambda - \beta^2}\, L = n\pi$, $n = 1, 2, 3,$ for non-trivial ϕ.

Thus $\lambda - \beta^2 = \left(\frac{n\pi}{L}\right)^2$, $\boxed{\lambda_n = \beta^2 + \left(\frac{n\pi}{L}\right)^2}$

4. Separation of variables in the PDE for a stretched string of non-uniform density leads to solutions of the form $u(x,t) = h(t)\phi(x)$ where and $h''(t) + \lambda h(t) = 0$ and

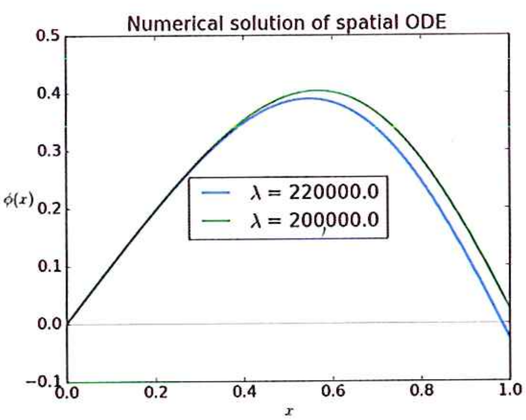
$$\phi''(x) + \lambda \overset{\rho(x)/T_0}{\frac{\cancel{T_0}}{\cancel{\rho(x)}}}\phi(x) = 0, \quad \phi(0) = 0, \quad \phi(L) = 0$$

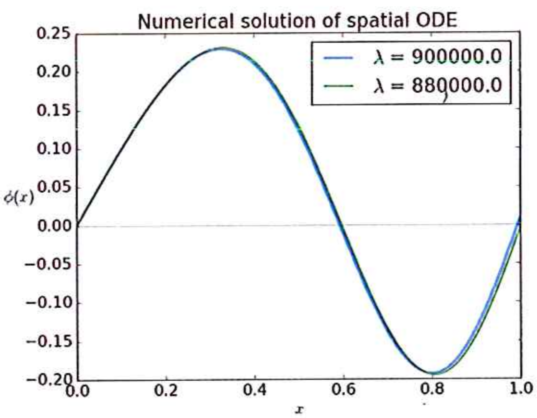
where T_0 is the constant positive tension, $\rho(x)$ is the positive density, and λ is the separation constant.

Solutions $\cos\sqrt{\lambda}\,t$ and $\sin\sqrt{\lambda}\,t$

Angular freq. is $\sqrt{\lambda}$.

The plots below show the numerical solution of the spatial ODE for a weird *nonuniform* guitar string of length 1m whose density varies linearly from 0.0005 kg/m at one end to 0.005 kg/m at the other, under a tension of 60N. In this system of units, time is measured in *seconds.* The initial conditions for the numerical solutions are $\phi(0) = 0$, $\phi'(0) = 1$.





(a) (2 points) Estimate the fundamental frequency of oscillation ν_1 (cycles per second) of this guitar string.

Estimate $\lambda_1 = 210{,}000$. Frequency $\nu_1 = \frac{\sqrt{\lambda_1}}{2\pi} \approx \frac{458}{2\pi} \approx 72.9$ Hz

(b) (2 points) Estimate the frequency ν_2 of the *second* mode.

Likewise $\lambda_2 \approx 890{,}000$. Frequency $\nu_2 = \frac{\sqrt{\lambda_2}}{2\pi} \approx \frac{943}{2\pi} \approx 150$ Hz

(c) (3 points) Contrast the ratio of the frequencies ν_2/ν_1 in parts (a) and (b) with the corresponding ratio for a *uniform* stretched string (which you may quote from memory or notesheet, or can re-derive if you have to).

For a uniform string, $\nu_2 = 2\nu_1$.

Here we have $\frac{\nu_2}{\nu_1} \approx 2.06$.

Hence for this non-uniform string the "first harmonic" is a little "sharp" i.e. ν_2 is a little more than one octave above the fundamental. (about half a semitone $= 2^{1/12}$).