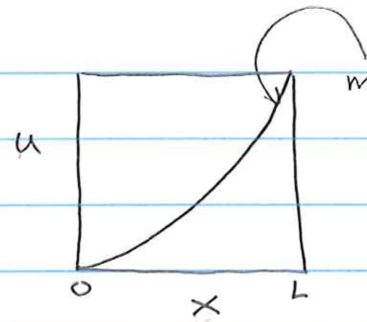
Temperature ramped up linearly at one end.

8.3.7

(a) 

my guess: concave up

The internal temperature lags behind what it would be if the current boundary condition were maintained for an extended period of time.

(b) With $u(0,t) = 0$, $u(L,t) = t$

we can set $\boxed{u(x,t) = \frac{x}{L}t + v(x,t)}$.

Then $\boxed{v(0,t) = 0}$ and $\boxed{v(L,t) = 0}$ $\forall t$.

As for the PDE, $u_t = \frac{x}{L} + v_t$

$u_{xx} = 0 + v_{xx}$

So $u_t = u_{xx}$ becomes

$\frac{x}{L} + v_t = v_{xx}$, or $\boxed{v_t = v_{xx} - \frac{x}{L}}$

As for the IC, $u(x,0) = \cancel{\frac{x}{L}0} + \boxed{v(x,0) = 0}$

(c) The eigenfunctions for the v-problem satisfy

$$\phi_{xx} = -\lambda\phi \quad \text{with} \quad \phi(0) = 0 = \phi(L).$$

Having solved this problem before, we know

eigenfunctions $\quad \lambda = \left(\frac{n\pi}{L}\right)^2, \quad n=1,2,3,\ldots \quad$ and $\quad \boxed{\phi_n(x) = \sin\frac{n\pi x}{L}}$

Writing $\quad v(x,t) = \sum_{n=1}^{\infty} a_n(t)\,\phi_n(x) = \sum_{n=1}^{\infty} a_n(t)\sin\frac{n\pi x}{L},$

we have $\quad v_t(x,t) = \sum_{n=1}^{\infty} a_n'(t)\,\phi_n(x) = \sum_{n=1}^{\infty} a_n'(t)\sin\frac{n\pi x}{L}$

and $\quad v_{xx}(t) = \sum_{n=1}^{\infty} a_n(t)\cdot\left(-\left(\frac{n\pi}{L}\right)^2\right)\sin\frac{n\pi x}{L}$

Thus the PDE $\quad v_t = v_{xx} - \frac{x}{L} \quad$ says

$$\sum_n a_n'(t)\sin\frac{n\pi x}{L} = -\sum_n \left(\frac{n\pi}{L}\right)^2 a_n(t)\sin\frac{n\pi x}{L} - \frac{x}{L}$$

or

$$\sum_{n=1}^{\infty}\left(a_n'(t) + \left(\frac{n\pi}{L}\right)^2 a_n(t)\right)\sin\frac{n\pi x}{L} = -\frac{x}{L}$$

Multiplying both sides by $\sin\frac{m\pi x}{L}$, integrating, and using the orthogonality of sines, we find

$$\left(a_m'(t) + \left(\frac{m\pi}{L}\right)^2 a_n(t)\right)\underbrace{\int_0^L \sin^2\frac{m\pi x}{L}\,dx}_{\frac{L}{2}} = -\frac{1}{L}\int_0^L x\sin\frac{m\pi x}{L}\,dx$$

$$a_m'(t) + \left(\frac{m\pi}{L}\right)^2 a_n(t) = -\frac{2}{L^2}\int_0^L x\sin\frac{m\pi x}{L}\,dx$$

Now $\int x \sin ax \, dx = -\frac{x}{a}\cos ax + \frac{\sin ax}{a^2}$ (table)

So $$\int_0^L x \sin \frac{m\pi x}{L} \, dx = -\frac{x}{\left(\frac{m\pi}{L}\right)} \cos \frac{m\pi x}{L} + \frac{\sin \frac{m\pi x}{L}}{\left(\frac{m\pi}{L}\right)^2} \Bigg|_0^L$$

$$= -\frac{L^2}{m\pi} \cos m\pi + 0 + 0 - 0$$

$$= -\frac{L^2}{m\pi}(-1)^m$$

Therefore

$$a_m'(t) + \left(\frac{m\pi}{L}\right)^2 a_m(t) = -\frac{2}{L^2}\left(\frac{-L^2(-1)^m}{m\pi}\right) = \frac{2(-1)^m}{m\pi}$$

The general solution of the corresponding homogeneous ODE is

$$a_n(t) = C e^{-\left(\frac{m\pi}{L}\right)^2 t}$$

and a particular solution of the nonhomogeneous eqn is

$$a_m(t) = \frac{2(-1)^m}{m\pi} \Big/ \left(\frac{m\pi}{L}\right)^2 = \frac{2L^2(-1)^m}{(m\pi)^3}$$

so that the general solution of the non-hom. eqn is

$$a_m(t) = C e^{-\left(\frac{m\pi}{L}\right)^2 t} + \frac{2L^2(-1)^m}{(m\pi)^3}$$

IC: Now we also have

$$\sum_{n=1}^{\infty} a_n(0) \sin \frac{n\pi x}{L} = 0 \leftarrow v(x,0)$$

By orthogonality of series, we have $a_n(0) = 0 \ \forall n$.

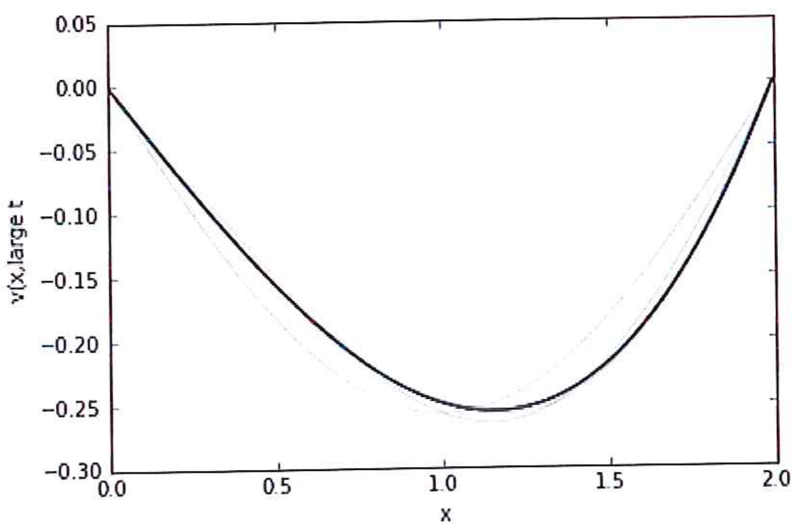
Therefore $C = -\frac{2L^2(-1)^m}{(m\pi)^3}$ and

$$* \longrightarrow \boxed{a_m(t) = \frac{2L^2(-1)^m}{(m\pi)^3}\left(1 - e^{-\left(\frac{m\pi}{L}\right)^2 t}\right)}$$ formula for time-varying coefficients.

(d) As $t \to \infty$ $\quad \boxed{a_n(t) \to \frac{2L^2(-1)^m}{(m\pi)^3}}$

```
%pylab inline
L = 2
def a(n): return (2*L**2/(pi**3))*(-1)**n/n**3 # your formula for a_n a
t infinite t here
def phi(n,x): return sin(n*pi*x/L) # your formula for phi_n(x) here
x = linspace(0,L,300)
v = zeros_like(x)
N = 20  # this is probably enough terms to get very close
for n in range(1,N):
     v += a(n)*phi(n,x)
     plot(x,v,'k',alpha=0.2)
ylabel('v(x,large t')
xlabel('x');
```

```
Populating the interactive namespace from numpy and matplot
lib
```



(e) The equation for equilibrium for v is

$$0 = v_{xx} - \frac{x}{L} \qquad \text{or} \qquad v_{xx} = \frac{x}{L}$$

This can be solved by integrating twice:

$$v_x = \frac{x^2}{2L} + c_1$$

$$v(x) = \frac{x^3}{6L} + c_1 x + c_2 .$$

Now the BC $v(0) = 0$ says $c_2 = 0$.
And the BC $v(L) = 0$ says

$$\frac{L^3}{6L} + c_1 L = 0$$

$$\frac{L}{6} + c_1 = 0 \rightarrow c_1 = -\frac{L}{6}$$

Thus $\boxed{v_E(x) = \frac{x^3}{6L} - \frac{Lx}{6}}$

(f) This should be <u>identical</u> to the ∞-time solution of the V PDE because heat problems relax to equilibrium.

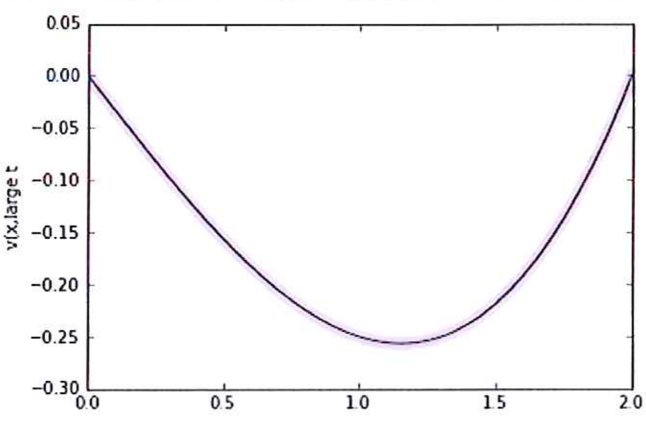
This is verified in the plot below where the two curves are seen to be coincident.

```
%pylab inline
L = 2
def a(n): return (2*L**2/(pi**3))*(-1)**n/n**3 # your formula for a_n at infinite t here
def phi(n,x): return sin(n*pi*x/L) # your formula for phi_n(x) here
x = linspace(0,L,300)
v = zeros_like(x)
N = 20  # this is probably enough terms to get very close
for n in range(1,N):
     v += a(n)*phi(n,x)
     plot(x,v,'k',alpha=0.2)
ylabel('v(x,large t')
xlabel('x');
```

```
# Compare with explicit equilibrium solution
veq = x**3/6/L-L*x/6
plot(x,veq,'m',lw=8,alpha=0.25)
plot(x,v,'k')
ylabel('v(x,large t')
xlabel('x');
```

magenta for V_E

black for large partial sum



(g) Set $w(x,t) = v(x,t) - v_E(x)$

$$= v(x,t) - \frac{x^3}{6L} + \frac{Lx}{6}$$

Then $w_t = v_t$, $w_{xx} = v_{xx} - \frac{x}{L}$

Therefore w satisfies the PDE

$$w_t = w_{xx} \qquad \text{(no source term for } w!\text{)}$$

and the BCs $w(0,t)=0$, $w(L,t)=0$.
We can just write down the solution of this problem (for general IC):

$$w(x,t) = \sum_{n=1}^{\infty} b_n e^{-\left(\frac{n\pi}{L}\right)^2 t} \sin\frac{n\pi x}{L}$$

The work for w is in finding the $\{b_n\}$ to satisfy the initial condition:

$$w(x,0) = -\frac{x^3}{6L} + \frac{Lx}{6}$$

Thus

$$\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} = -\frac{x^3}{6L} + \frac{Lx}{6}$$

By orthog of sines,

$$b_n = \frac{2}{L}\int_0^L \left(-\frac{x^3}{6L} + \frac{Lx}{6}\right)\sin\frac{n\pi x}{L}\,dx = \frac{-2(-1)^n L^2}{(n\pi)^3}$$

```
from sympy import *
init_printing()
x,L = symbols('x,L')
n = symbols('n',integer=True,positive=True)
bn = 2/L*integrate((-x**3/6/L +L*x/6)*sin(n*pi*x/L),(x,0,L))
bn
```

$$-\frac{2(-1)^n L^2}{\pi^3 n^3}$$

Thus $w(x,t) = \sum_{n=1}^{\infty} \frac{-2(-1)^n L^2}{(n\pi)^3} e^{-\left(\frac{n\pi}{L}\right)^2 t} \sin\frac{n\pi x}{L}$

and we get

$$u(x,t) = w(x,t) + v_E(x,t) + \frac{xt}{L}$$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \frac{-2(-1)^n L^2}{(n\pi)^3} e^{-\left(\frac{n\pi}{L}\right)^2 t} \sin\frac{n\pi x}{L} + \frac{x^3}{6L} - \frac{Lx}{6} + \frac{xt}{L}}$$

(h) Well, it made the PDE homogeneous and one that we've solved many times.
On the other hand, the hard work just migrated to the IC.

The alternative was to go to $u(x,t)$ directly from the solution for $v(x,t)$:

$$v(x,t) = \sum_{n=1}^{\infty} \frac{2L^2(-1)^n}{(n\pi)^3}\left(1 - e^{-\left(\frac{n\pi}{L}\right)^2 t}\right) \sin\frac{n\pi x}{L}.$$

and so

$$\boxed{u(x,t) = \frac{xt}{L} + \sum_{n=1}^{\infty} \frac{2L^2(-1)^n}{(n\pi)^3}\left(1 - e^{-\left(\frac{n\pi}{L}\right)^2 t}\right) \sin\frac{n\pi x}{L}}$$

I would say that without technology, using w is a bit harder because you have to integrate $x^3 \sin\frac{n\pi x}{L}$. But the formula for u is easier to interpret.