9.3.9 $u'' + u = f$ by variation of parameters

$u(0) = 0, \quad u(L) = 0$

Corresponding homogeneous eqn is

$u'' + u = 0$

which has general solution

$u(x) = a\cos x + b\sin x$

or, more conveniently

$u(x) = A\sin(x - \phi)$.

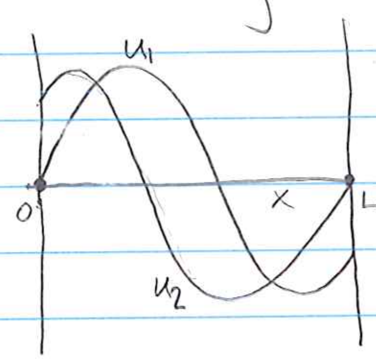
This form is convenient for making it match the BCs.

Let us choose

$u_1(x) = \sin x$, which satisfies $u(0) = 0$

and $u_2(x) = \sin(x - L)$, which satisfies $u(L) = 0$

(They look something like this:



Note that the condition $L \neq n\pi$ guarantees $u_1(L) \neq 0$, and $u_2(0) \neq 0$.)

Then we look for a solution of the non-homogeneous equation of the form

$$u(x) = v_1(x)\sin x + v_2(x)\sin(x - L).$$

Then

$$u'(x) = v_1'(x)\sin x + v_1(x)\cos x + v_2'(x)\sin(x-L) + v_2(x)\cos(x-L)$$

Now let's "spend" one of our degrees of freedom to make $\boxed{v_1(x)\sin x + v_2'(x)\sin(x-L) = 0}$

Then $u'(x) = v_1(x)\cos x + v_2(x)\cos(x-L)$

and

$$u''(x) = v_1'(x)\cos x + v_1(x)(-\sin x) + v_2'(x)\cos(x-L) + v_2(x)(-\sin(x-L))$$

and

$$u''(x) + u(x) = v_1'(x)\cos x - \cancel{v_1(x)\sin x} + v_2'(x)\cos(x-L) - \cancel{v_2(x)\sin(x-L)} + \cancel{v_1(x)\sin x} + \cancel{v_2(x)\sin(x-L)}$$

$$= v_1'(x)\cos x + v_2'(x)\cos(x-L).$$

This must be $f(x)$, so in all we have

$$v_1'(x)\cos x + v_2'(x)\cos(x-L) = f(x) \qquad (*)$$

from previous page $\longrightarrow$ $$v_1'(x)\sin x + v_2'(x)\sin(x-L) = 0$$

Multiplying the first eqn by $\sin(x-L)$ and the second eqn by $\cos(x-L)$, and subtracting, we eliminate v_2' and obtain

$$v_1'(x)\left(\cos x\sin(x-L) - \sin x\cos(x-L)\right) = f(x)\sin(x-L)$$

$$v_1'(x) = f(x)\sin(x-L) \Big/ \left(\cos x \sin x-L - \sin x\cos(x-L)\right)$$

Can we simplify that denominator? Thankfully, yes! It is $-\sin L$. So

```
from sympy import *

x,L = symbols('x,L',real=True)

simplify( cos(x)*sin(x-L)-sin(x)*cos(x-L) )

-sin(L)
```

$$\boxed{v_1'(x) = -\frac{f(x)\sin(x-L)}{\sin L}}$$

From the second equation (*), we have

$$v_2'(x) = \frac{-v_1(x)\sin x}{\sin(x-L)},$$

so

$$v_2'(x) = -\left(\frac{-f(x)\cancel{\sin(x-L)}}{\sin L}\right)\frac{\sin x}{\cancel{\sin(x-L)}}$$

$$\boxed{v_2'(x) = \frac{f(x)\sin x}{\sin L}}$$

To obtain $v_1(x)$, $v_2(x)$ we now just need to integrate: (using x_0 as the "dummy" integration variable)

$$v_1(x) = \int_0^x \frac{-f(x_0)\sin(x_0-L)}{\sin L}\,dx_0 + c_1$$

$$v_2(x) = \int_0^x \frac{f(x_0)\sin x_0}{\sin L}\,dx_0 + c_2$$

To determine the constants c_1, c_2, we apply the BCs:

- $u(0) = v_1(0)\underbrace{u_1(0)}_{0} + v_2(0)u_2(0) = 0$

$$c_2 \cdot \sin(-L) = 0 \Rightarrow \boxed{c_2 = 0}$$

(again using $L \neq n\pi$, so $\sin L \neq 0$)

- $u(L) = v_1(L)u_1(L) + v_2(L)\underbrace{u_2(L)}_{0} = 0$

$$\left(\int_0^L \frac{-f(x_0)\sin(x_0-L)}{\sin L}\,dx_0 + c_1\right)\sin L = 0$$

Again since we're guaranteed $\sin L \neq 0$,

$$c_1 = \int_0^L \frac{f(x_0)\sin(x_0-L)}{\sin L}\,dx_0$$

Thus

$$v_1(x) = \int_0^x \frac{-f(x_0)\sin(x_0-L)}{\sin L}\,dx_0 + \int_0^L \frac{f(x_0)\sin(x_0-L)}{\sin L}\,dx_0$$

$$= \int_x^L \frac{f(x_0)\sin(x_0-L)}{\sin L}\,dx_0$$

and the solution of the non-homog. eqn is thus

$$u(x) = \left(\int_x^L \frac{f(x_0)\sin(x_0-L)}{\sin L}\,dx_0\right)\cdot \sin x + \left(\int_0^x \frac{f(x_0)\sin x_0}{\sin L}\,dx_0\right)\cdot \sin(x-L)$$

$$= \int_0^x f(x_0)\left[\frac{\sin x_0 \cdot \sin(x-L)}{\sin L}\right]dx_0 + \int_x^L f(x_0)\left[\frac{\sin(x_0-L)\sin x}{\sin L}\right]dx_0$$

Which we can write as $u(x) = \displaystyle\int_0^L f(x_0)\,G(x,x_0)\,dx_0$

if we set

$$\boxed{G(x,x_0) = \begin{cases} \dfrac{\sin x_0 \cdot \sin(x-L)}{\sin L}, & x_0 < x \\[2ex] \dfrac{\sin(x_0-L)\sin x}{\sin L}, & x_0 > x \end{cases}}$$

This is the Green's function for the problem.

9.3.9

5

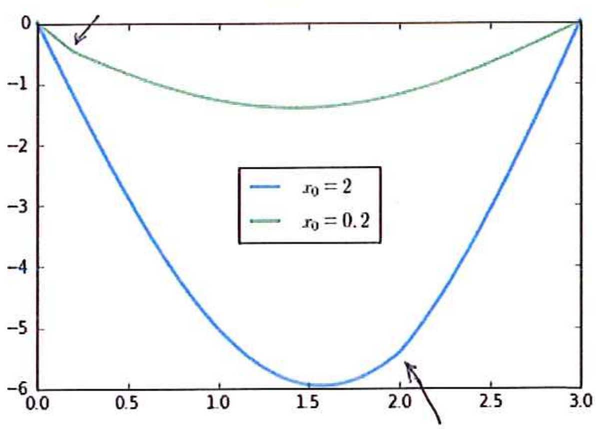
(x) Plot of Green's function

9.3.9 Green's function for ODE BVP

```
%pylab inline
from numpy import *
def G(x,x0):
    Gvals       = sin(x0-L)*sin(x )/sin(L)
    xgtx0 = x > x0
    Gvals[xgtx0] = sin(x[xgtx0] -L)*sin(x0)/sin(L)
    return Gvals
```

```
Populating the interactive namespace from numpy and matplotlib
```

```
L = 3
x = linspace(0,L,300)
for x0 in [2,.2]:
    plot(x,G(x,x0),label='$x_0='+str(x0)+'$',lw=2,alpha=0.5,clip_on=False)
legend(loc='center');
```



discontinuity in slope