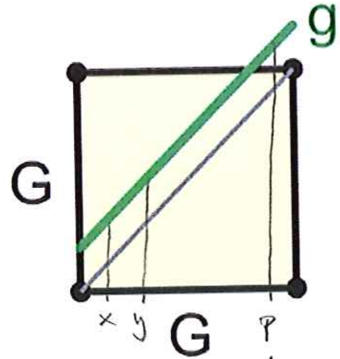
Hw3

1.

For each of the 6 functions, g, whose graphs are sketched below:

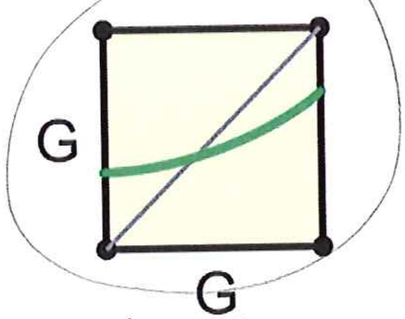
- a • If g does not have a unique fixed point in G, state a hypothesis of the Contraction Mapping Theorem that is not satisfied by g.
- b • If g does have a unique fixed point in G but iteration of g (starting in G) does not certainly converge to that fixed point, state a hypothesis of the CM theorem that is not satisfied by g.
- c • If g is not a contraction, mark 2 points x, y such that $|g(x) - g(y)| \geq |x - y|$
- d • If g is does not map G into G, mark a point p in G such that g(p) is not in G.
- e • If iteration of g converges to a unique fixed point despite g not satisfying the hypotheses of the CM Theorem, circle the picture.

People came up with very varied and nice ways of organizing their answers to this question Better than mine!

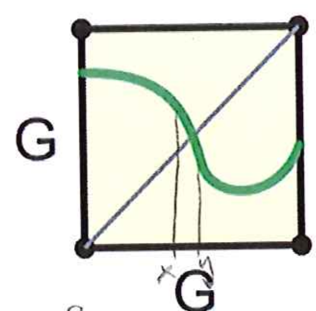


a
No fixed point.
$g: G \not\to G$
Also not contraction
if slope = 1.

(not supposed to be circled)

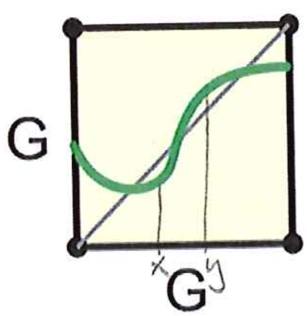


Unique f.p.
Is a contraction
$g: G \to G$



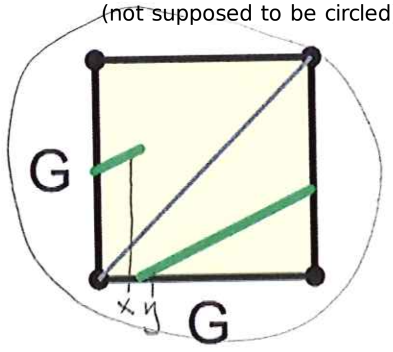
b
∃ unique f.p.
g not a contraction

iteration does not converge to f.p.
because |g'| at f.p. is > 1

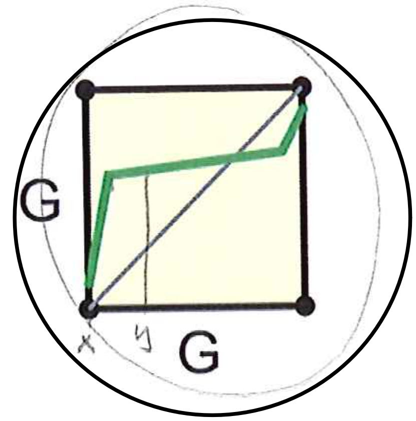


non-unique f.p.
not a contraction
g:G into G

(not supposed to be circled)



not even 1 f.p.
not a contraction



unique fp
convergence to fp
from any x in G
not a contraction

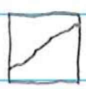
HW3

2. $g(x) = \dfrac{x^2+6}{5}$ on $[1, 2.3] = G$.

(a) Show g maps G into G

I will show g is (strictly) increasing on G.
$g'(x) = \dfrac{2x}{5}$. Thus $g'(x) > 0$ for $x > 0$.

Since $x > 0$ $\forall x$ in G, $g' > 0$ on G.
Therefore g is strictly increasing on G.
This means $g(1) \leq g(x) \leq g(2.3)$ $\forall x \in G$.
Then observe $g(1) = \dfrac{1+6}{5} = \dfrac{7}{5} \in G$

G 
G

$$\begin{array}{r} 2.3 \\ \times\, 2.3 \\ \hline .69 \\ 4.6 \\ \hline 5.29 \end{array}$$

and $g(2.3) = \dfrac{(2.3)^2+6}{5} = \dfrac{5.29+6}{5} = \dfrac{11.29}{5} = 2.258$ $\in G$.

We conclude $g(x) \in G$ $\forall x \in G$. QED.

(b) Show g is a contraction on G.
We need to show g is Lipschitz on G with $L < 1$.
We have Prop 2.1 (p41) which provides this if
g is differentiable on G with $|g'| \leq$ some $L < 1$.

We have $|g'(x)| = \left|\dfrac{2x}{5}\right|$. $\max\limits_{x \in G} |x| = 2.3$,

so $|g'(x)| \leq \dfrac{2 \cdot 2.3}{5} = \dfrac{4.6}{5} < 1$ on G.

Thus g is a contraction on G.

HW3

3

Prop 2.3 Let $\rho > 0$ and $G = [c-\rho, c+\rho]$.
If g is a contraction on G with Lipschitz const L in $[0,1)$, and $|g(c) - c| \leq (1-L)\rho$ then g maps G into itself.

Proof. Take $x \in G$. Explanations highlighted.
(We need to show $g(x) \in G$.) Def. of g maps G into G.
Then

$$|g(x) - c| = |g(x) - g(c) + g(c) - c|$$

Identity: $-g(c) + g(c) = 0$.

$$\leq |g(x) - g(c)| + |g(c) - c|$$

Triangle inequality.

$$\leq L|x - c| + (1-L)\rho$$

First term from Lipschitz property
Second term from other hypothesis

$$\leq L\rho + (1-L)\rho$$

$x \in G = [c-\rho, c+\rho] \Rightarrow |x-c| \leq \rho$

$$= \rho$$

Algebraic simplification.

The hypothesis

$$|g(c) - c| \leq (1 - L)\rho$$

is requiring that g(c) is not too close to the boundary of G, given the Lipschitz constant, L.

Specifically, it's saying this Lipschitz cone at c doesn't go outside of G
(extreme case shown) :