MTH 538

Midterm Exam 1

Thursday, March 10, 2022

Name: SOLUTIONS

UB Person #: ____________________

For maximum credit, show your work.

To gain 4 points for "style",

your answers should be easy to read and easy to follow.

All answers must be **written in the space provided** for each question.
Scratch work can be done on the the backs of the pages.
Write answers only on the **front sides** of the pages.
Only the fronts will be scanned and graded.

Turn in all pages, whether or not you have written on them.

The use of handwritten notes on one copy of the official notesheet is allowed.

1. Taylor series methods

Consider solutions $y(t)$ of the equation $y'' + 3y' + t^2 y^2 = 5$.

If $y(1) = 3$ and $y'(1) = 7$, find an approximation to $y(1+h)$ whose error is $O(h^4)$.

Given $\begin{cases} y(1) = 3 \\ y'(1) = 7 \end{cases}$

DE: $y'' = 5 - 3y' - t^2 y^2 \rightarrow y''(1) = 5 - 3\cdot 7 - 1^2\cdot 3^2$

$= 5 - 21 - 9 = -25$

Differentiating:

$y''' = 0 - 3y'' - 2ty^2 - t^2\cdot 2yy' \rightarrow y'''(1) = -3\cdot(-25) - 2\cdot 1\cdot 3^2 - 1^2\cdot 2\cdot 3\cdot 7$

$= +75 - 18 - 42 = 15$

Thus by Taylor's theorem

$$y(1+h) = y(1) + hy'(1) + \frac{h^2}{2}y''(1) + \frac{h^3}{3!}y'''(1) + O(h^4)$$

$$= \boxed{3 + 7h - \frac{25}{2}h^2 + \frac{15}{6}h^3 + O(h^4)}$$

2. Linear multistep methods

(a) Write down the equations that must be satisfied by the coefficients $\{\alpha_i\}$, $\{\beta_i\}$ of a linear multistep method in order for it to be a pth order method.

$$\sum_{\ell=0}^{k} \alpha_\ell = 0$$

$$\sum_{\ell=0}^{k} \frac{\ell^q \alpha_\ell}{q!} - \frac{\ell^{q-1}}{(q-1)!}\beta_\ell \quad , \; q = 1, \dots p$$

(b) Specialize your answer in part to the 2-step Backward Differentiation method, which has $\beta_0 = 0$, and $\beta_1 = 0$. Write out the 3 equations that must be satisfied for the method to have 2nd order accuracy - explicitly, with no Σ symbols.

α_0	α_1	1
0	0	β_2

α_2 wolog

$$\alpha_0 + \alpha_1 + 1 = 0 \rightarrow \boxed{\alpha_0 + \alpha_1 = -1}$$

$q = 1$:
$$\frac{0\alpha_0 + 1\alpha_1 + 2 \cdot \overset{\alpha_2}{1}}{1!} - \frac{\overset{1}{2^0}\beta_2}{0!} = 0$$

$$\boxed{\alpha_1 - \beta_2 = -2}$$

$q = 2$:
$$\frac{0^2\alpha_0 + 1^2\alpha_1 + 2^2 \cdot 1}{2!} - \frac{2^1\beta_2}{1!} = 0$$

mult. by $2! = 2$:

$$\boxed{\alpha_1 - 4\beta_2 = -4}$$

(c) Solve your equations in part (b) for the coefficients.

1 $\boxed{\alpha_0 + \alpha_1 = -1}$

2 $\boxed{\alpha_1 - \beta_2 = -2}$

3 $\boxed{\alpha_1 - 4\beta_2 = -4}$

Subtracting 3 from 2 :

$+3\beta_2 = 2 \longrightarrow$

Then 2 $\rightarrow$ $\alpha_1 = -2 + \frac{2}{3} \longrightarrow$

and then 1 $\Rightarrow$ $\alpha_0 = -1 - \left(\frac{-4}{3}\right) \longrightarrow$

$$\boxed{\begin{aligned} \beta_2 &= \frac{2}{3} \\ \alpha_1 &= \frac{-4}{3} \\ \alpha_0 &= \frac{1}{3} \end{aligned}}$$

(So formula is

$$\frac{1}{3}y_j - \frac{4}{3}y_{j+1} + y_{j+2} = \frac{2}{3}hf_{j+2}$$

or $$y_{j+2} = \frac{-1}{3}y_j + \frac{4}{3}y_{j+1} + \frac{2}{3}hf_{j+2} \, .)$$

3. Stability

(a) Find the characteristic polynomial in the variable z for the linear 2-step method with $\alpha_0 = 0$, $\alpha_1 = -1$, $\beta_0 = -\frac{1}{2}$, $\beta_1 = \frac{3}{2}$, applied to the scalar DE $y' = \lambda y$.

$$\alpha_0 \qquad -\alpha_1 \qquad \beta_1 \qquad \beta_0$$

$$y_{j+2} = \cancel{0\, y_j} + 1\, y_{j+1} + h\left(\tfrac{3}{2} f(y_{j+1}) - \tfrac{1}{2} f(y_j)\right)$$

$$= y_{j+1} + h\lambda\left(\tfrac{3}{2} y_{j+1} - \tfrac{1}{2} y_j\right) \quad \text{if } f(y) = \lambda y$$

$$= -\tfrac{1}{2} h\lambda\, y_j + \left(1 + \tfrac{3}{2} h\lambda\right) y_{j+1}$$

So $$\begin{bmatrix} y_{j+1} \\ y_{j+2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{2}h\lambda & 1 + \frac{3}{2}h\lambda \end{bmatrix} \begin{bmatrix} y_j \\ y_{j+1} \end{bmatrix}$$

Char poly is $$\det \begin{bmatrix} -z & 1 \\ -\frac{1}{2}h\lambda & 1 + \frac{3}{2}h\lambda - z \end{bmatrix} = \boxed{(-z)\left(1 + \tfrac{3}{2}h\lambda - z\right) + \tfrac{1}{2}h\lambda}.$$

(b) For the method of part (a) and $\lambda < 0$, as h is increased from 0 we leave the region of absolute stability as an eigenvalue (one of the roots of the polynomial in (d)) passes through -1. At which $h\lambda$ does this happen?

If $z = -1$, then characteristic eqn is

$$-(-1)\left(1 + \tfrac{3}{2}h\lambda - (-1)\right) + \tfrac{1}{2}h\lambda = 0$$

$$2 + \tfrac{3}{2}h\lambda + \tfrac{1}{2}h\lambda = 0$$

$$2h\lambda = -2$$

$$\boxed{h\lambda = -1}$$

(c) Write down a (useless) explicit 1-step method that is stable but not consistent. Use the standard notation y_j, y_{j+1}, f, t, h. Briefly say why it's stable and why it's not consistent.

$$y_{j+1} = \tfrac{1}{10} y_j + 5hf(t_j, y_j)$$

Stable because for $f \equiv 0$, $y_{j+1} = \frac{1}{10} y_j$ has eigenvalue $\frac{1}{10} \le 1$.

Not consistent because $\Phi\big|_{h=0} = 5f \neq f$.

4. Actual use

Take one Euler's method step of size $h = 0.2$ applied to $y' = -3y$, $y(0) = 10$.

$t_0 = 0$, $y_0 = 10$, $h = .2$, $f(t,y) = -3y$

$$y_1 = y_0 + hf(t_0, y_0) \quad \text{(Euler)}$$
$$= 10 + 0.2 \cdot (-3 \cdot 10)$$
$$= 10 - 6 = \boxed{4}$$

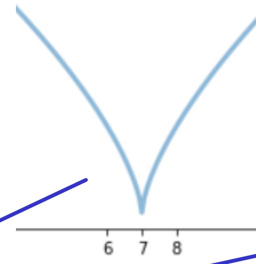
Then take one step of the Adams-Bashforth 2-step, which is the method in 3(a).

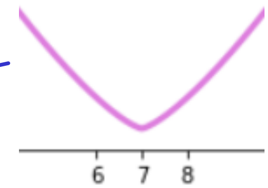
$$y_2 = y_1 + \left(-\tfrac{1}{2}\right) hf(t_0, y_0) + \tfrac{3}{2} hf(t_1, y_1) \quad \text{(AB2)}$$
$$= 4 + \left(-\tfrac{1}{2}\right)(0.2)(-3 \cdot 10) + \tfrac{3}{2}(0.2)(-3 \cdot 4)$$
$$= 4 + 3 - (0.1) \cdot 36$$
$$= 7 - 3.6$$
$$= \boxed{3.4}$$

5. Existence, uniqueness, extent, smoothness

(a) Consider the four differential equations

$$(i)\quad \frac{dy}{dt} = p(t)y,$$
$$(ii)\quad \frac{dy}{dt} = p(t)y^2,$$
$$(iii)\quad \frac{dy}{dt} = p(t)|y-7|^{2/3},$$
$$(iv)\quad \frac{dy}{dt} = p(t)|y-7|^{4/3},$$





where $p : \mathbb{R} \to \mathbb{R}$ is a continuous function.

Say the most you can about existence, uniqueness, and extent on the t-axis of solutions of each of the differential equations with the initial condition $y(0) = 7$. You may refer to Theorems "7.1" and "7.2" from the textbook.

(i) $f(t,y) = p(t)y$. This is continuous and Lipschitz in y
so Theorem 7.2 applies: unique solution extending to all of $\mathbb{R}$.

(ii) $f(t,y) = p(t)y^2$. This is continuous but not Lipschitz in y,
so only Theorem 7.1 applies: unique solution guaranteed only locally.

(iii) $f(t,y) = p(t)|y-7|^{2/3}$. This is continuous but does not have
continuous y derivative at $y=7$. Neither theorem applies,
so no guarantee even of local uniqueness.
There is a solution $y(t)=7$ that extends to all of $\mathbb{R}$
but it is in fact not the only one that satisfies $y(0)=7$.

(iv) $f(t,y) = p(t)|y-7|^{4/3}$. This is continuous with continuous
y-derivative at $y=7$ so Theorem 7.1 guarantees a unique solution
locally. In fact, $y(t)=7$ is a solution that extends to all
of $\mathbb{R}$ and since Theorem 7.1 applies at every t, this solution
is the unique solution of the IVP.

(b) Theorem 7.1 tells us a unique solution of the IVP in Question 1 exists for all sufficiently small h, but the error estimate assumes some smoothness of that solution. Briefly explain why such an assumption is justified.

Theorem 7.1 implies local unique solution exists.
DE implies y'' exists, so y, y' are differentiable.
Then y'' as a linear combination of differentiable functions
is itself differentiable.
By differentiating the DE twice, applying the same logic,
$y^{(4)}$ is differentiable, hence continuous so the requirements of Taylor's theorem are met.