1(a) $y'' = y' - y \qquad y(3) = 3 \qquad y'(5) = 0$

DE $\boxed{\begin{aligned} y' &= u \\ u' &= u - y \end{aligned}}$ BCs $y(3) = 3$

$u(5) = 0$

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(b) $\begin{bmatrix} y_1 \\ u_1 \end{bmatrix} = \begin{bmatrix} y_0 \\ u_0 \end{bmatrix} + h \begin{bmatrix} u_0 \\ u_0 - y_0 \end{bmatrix}$

With $h = 2$, $y_0 = 3$, $u_0 = 0$ (my 1st choice)

$$\begin{bmatrix} y_1 \\ u_1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 0-3 \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \end{bmatrix}$$

u_0	u_1
2	0
1	-3
0	-6
-1	-9

With $u_0 = 1$ (my 2nd choice)

$$\begin{bmatrix} y_1 \\ u_1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 1-3 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$$

Because u_1 is affine in u_0,

For grading:
$u_0 = -1$
$\begin{bmatrix} 3 \\ -1 \end{bmatrix} + 2\begin{bmatrix} -1 \\ -1-3 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \end{bmatrix}$

u_0	u_1
$0 \to$	-6
$1 \to$	-3
$2 \to$	0 .

u_0	u_1
3	3
4	6

$$\begin{bmatrix} u_1 \\ y_1 \end{bmatrix}\Bigg|_{u_0=2} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 2-3 \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \end{bmatrix}$$

Answer $y(5) \approx 7$.

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1 2 3 4

2. Let $A_i = A(t_i)$, $f_i = f(t_i)$.

$$y_1 = y_0 + hA_0y_0 + hf_0 = (I + hA_0)y_0 + hf_0$$

$$y_2 = y_1 + hA_1y_1 + hf_1 = (I + hA_1)y_1 + hf_1$$

$$= (I + hA_1)\left[(I+hA_0)y_0 + hf_0\right] + hf_1$$

$$= (I + hA_1)(I + hA_0)y_0 + (I + hA_1)hf_0 + hf_1$$

$$y_3 = y_2 + hA_2y_2 + hf_2 = (I + hA_2)y_2 + hf_2$$

$$= (I + hA_2)\left[(I+hA_1)(I+hA_0)y_0\right] + (I+hA_2)\left[(I+hA_1)hf_0 + hf_1\right] + hf_2$$

$$y_n = \underbrace{\prod_{i=0}^{n-1}(I + hA_i)y_0}_{= \text{"}M_n\text{"}} + \underbrace{\sum_{i=1}^{n-1}\left(\prod_{j=i+1}^{n-1}(I + hA_j)\right)hf_i}_{= \text{"}b_n\text{"}}$$

Thus

$$y_n = M_n y(0) + b_n$$

where neither M_n nor b_n depends on $y(0)$.

3 $\quad \left((1+x)y' \right)' + y = 3x^2 + 9x + 3$

$$(1+x)'y' + (1+x)y'' + y = 3x^2 + 9x + 3$$

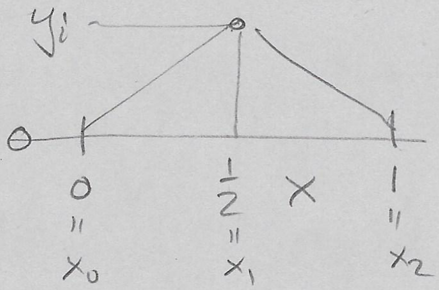
$$(1+x)y'' + y' + y = 3x^2 + 9x + 3$$

3(a) FD.

Use $\quad y'(x_i) \approx \dfrac{y(x_{i+1}) - y(x_{i-1})}{h}$

$$y''(x_i) \approx \frac{y(x_{i+1}) - 2y(x_i) + y(x_{i-1})}{h^2}$$

In this case



$$y'(x_1) \approx \frac{0 - 0}{\frac{1}{2}} = 0$$

$$y''(x_1) \approx \frac{0 - 2y_1 + 0}{\left(\frac{1}{2}\right)^2} = -8y_1$$

We want

$$\left(1 + \tfrac{1}{2}\right)(-8y_1) + 0 + y_1 = 3 \cdot \tfrac{1}{4} + 9 \cdot \tfrac{1}{2} + 3$$

$$\underbrace{(-12 + 1)}_{-11} y_1 = \frac{3 + 18 + 12}{4} = \frac{33}{4}$$

$$y_1 = -\frac{3}{4}.$$

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3(b) collocation at $x=\frac{1}{2}$ with $\phi(x) = x(1-x)$.

I.e. $Y(x) = a\,x(1-x)$ with

$$\left(1+\tfrac{1}{2}\right)Y''\left(\tfrac{1}{2}\right) + Y'\left(\tfrac{1}{2}\right) + Y\left(\tfrac{1}{2}\right) = \frac{33}{4} \leftarrow \text{from (a).}$$

$$Y'(x) = a(1-2x), \quad Y''(x) = a(-2).$$

So want

$$\frac{3}{2}(-2a) + a\underbrace{\left(1-2\cdot\tfrac{1}{2}\right)}_{0} + a\cdot\tfrac{1}{2}\left(1-\tfrac{1}{2}\right) = \frac{33}{4}$$

$$a\left[-3+\frac{1}{4}\right] = 33/4$$

$$a = \frac{33}{4}\Big/ -\frac{11}{4} = -3$$

$$Y_c(x) = -3x(1-x).$$

1:52 $\quad Y_c\left(\tfrac{1}{2}\right) = -3\cdot\tfrac{1}{2}\cdot\tfrac{1}{2} = -\frac{3}{4}.$

3c

$$((1+x)y')' + y = 3x^2+9x+3$$

$r(x) = -(1+x)$, $s(x)=1$, $f(x) = 3x^2+9x+3$

$\phi(x) = x(1-x)$, $Y = a\phi$

$\phi'(x) = 1-2x$, $Y' = a\phi'$

Galerkin : $\int_0^1 r(a\phi')\phi' + s(a\phi)\phi = \int_0^1 f\phi$

$$a \cdot \int_0^1 -(1+x)(1-2x)^2 + 1\cdot(x(1-x))^2 \, dx = \int_0^1 (3x^2+9x+3)x(1-x)\, dx$$

$1-4x+4x^2$

$$a = \frac{\int_0^1 3x^3+9x^2+3x-3x^4-9x^3-3x^2 \, dx}{\int_0^1 -1+4x-4x^2-x+4x^2-4x^3+x^2+2x^3+x^4 \, dx}$$

$$= \frac{\int_0^1 -3x^4-6x^3+6x^2+3x \, dx}{\int_0^1 -1+3x-4x^3+x^2-2x^3+x^4 \, dx}$$

$$= \frac{3\int_0^1 -x^4-2x^3+2x^2+x \, dx}{\int_0^1 -1+3x+x^2-6x^3+x^4 \, dx}$$

$$= 3\left[-\frac{1}{5}-\cancel{\frac{2}{4}}+\frac{2}{3}+\cancel{\frac{1}{2}}\right] \Big/ \left[-1+\cancel{\frac{3}{2}}+\frac{1}{3}-\cancel{\frac{6}{4}}+\frac{1}{5}\right]$$

$$= 3\left[\frac{-3+10}{15}\right] \Big/ \left[\frac{-15+5+3}{15}\right] = 3\cdot\frac{7/15}{-7/15} = -3 .$$

Galerkin approx $Y_G(x) = -3x(1-x)$.

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4 $f(0) = f(0)$

$f(h) = f(0) + hf'(0) + \frac{h^2}{2}f''(0) + O(h^3)$

$f(2h) = f(0) + 2hf'(0) + \frac{(2h)^2}{2}f''(0) + O(h^3)$

Seek c_0, c_1, c_2 such that

$c_0 f(0) + c_1 f(h) + c_2 f(2h) = f'(0) + O(h^2)$

Thus require

$c_0 + c_1 + c_2 = 0$

$c_0 \cdot 0 + c_1 \cdot h + c_2 \cdot 2h = 1$

$c_0 \cdot 0 + c_1 \frac{h^2}{2} + c_2 \cdot \frac{(2h)^2}{2} = 0$

i.e. $c_0 + c_1 + c_2 = 0$ ①

$c_1 + 2c_2 = \frac{1}{h}$ ②

$c_1 + 4c_2 = 0$ ③

③ − ② : $2c_2 = -\frac{1}{h}$, $c_2 = -\frac{1}{2h}$

② $\longrightarrow$ $c_1 = \frac{1}{h} - 2 \cdot \left(-\frac{1}{2h}\right)$

$= \frac{1}{h}(1+1) = \frac{2}{h}$

① $\longrightarrow$ $c_0 = -c_1 - c_2$

$= -\left(\frac{2}{h}\right) - \left(-\frac{1}{2h}\right) = \frac{-2+\frac{1}{2}}{h} = -\frac{3}{2h}$

So $f'(0) = \frac{-\frac{3}{2}f(0) + 2f(h) - \frac{1}{2}f(2h)}{h} + \underbrace{\frac{O(h^3)}{h}}_{= O(h^2)}$

or $= \frac{-3f(0) + 4f(h) - 1f(2h)}{2h}$.

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