MTH 538 Spring 2022 Final Exam Questions

Tuesday May 17

1. A method we might call *centered Euler*

Consider the linear multistep formula $y_{j+2} = y_j + 2hf_{j+1}$ for approximating the solution of the ODE IVP $y' = f(t, y)$, $y(t_0) = y_0$, where f_j means $f(t_j, y_j)$.

(a) Determine the *order of accuracy* of the method, and from that conclude whether the method is *consistent* or not.

(b) Determine if the method is *stable* or not. Recall that stable means there are no growing modes when $f \equiv 0$.

(c) From your answers for (a) and (b) conclude whether the method is *convergent* or not, justifying your answer.

(d) Show that when applied to $y' = \lambda y$, the method blows up even for small negative $h\lambda$.

(e) Does the fact stated in part (d) mean the method is useless? Justify your answer.

2. Reasons to prefer one ODE IVP method over another

(a) Name or describe a method you like (other than Euler's!) for approximating the solution of an ODE IVP, and give a good reason you like it.

(b) State one way in which the 4th order Adams-Bashforth multi-step method is better than a 4th order Runge-Kutta multi-stage method.

(c) Same as part (b) with the methods reversed.

(d) A 4D autonomous system of ODE has linearization with a pair of eigenvalues around $-1 \pm i$ and another around -10^6. Name or specify concretely a method that would be appropriate for solving an IVP for this system, and explain why.

3. Absolute stability of the Taylor series method

(a) Determine the eigenvalue, z, of the nth order Taylor series method with step size h applied to the scalar equation $y' = \lambda y$.

(b) Define *absolute stability* of a method, and using your answer to part (a), argue that for any n, the method is absolutely stable for all sufficiently small negative real $h\lambda$.

(c) At which negative real $h\lambda$ do the 1st order (Euler) and 2nd order methods lose absolute stability?

(d) Is the 3rd order method absolutely stable at the $h\lambda$ values you computed in part (c)? Justify your answer.

4. 2-stage Runge-Kutta method

(a) A 2-stage Runge-Kutta method for generating approximations $y^{[k]} \approx y(kh)$, $k = 0, 1, 2, \ldots$, to the solution of an IVP for the autonomous system $y' = f(y)$ employs a formula of the form

$$\widetilde{y}^{[k+1]} = y^{[k]} + haf(y^{[k]}),$$
$$y^{[k+1]} = y^{[k]} + h\left(bf(y^{[k]}) + cf(\widetilde{y}^{[k+1]})\right),$$

where a, b, c are constants. Determine necessary and sufficient conditions on a, b, c for the method to have local truncation error $O(h^3)$, and write the general such method in terms of just c.

You may assume that f has partial derivatives to at least third order on the relevant region so as to justify the use of Taylor's theorem. Specialize to $y(t) \in \mathbb{R}^2$ to keep the calculation explicit yet manageable.

(b) Explain why in general we can consider only autonomous systems of ODEs, if we wish, without loss of generality.

5. Dependence of the solution of an ODE IVP on a parameter

[Answer this **OR** 6 **OR** 7.]

Consider an IVP for an autonomous scalar ODE that depends on a (t-independent) parameter, p:

$$\frac{\partial y(t,p)}{\partial t} = f(y,p), \qquad y(0,p) = y_0 \qquad (1)$$

where we wish to know the derivative of the solution at time t with respect to the parameter p.

(a) Call the derivative we are interested in knowing $v(t,p)$:

$$v(t,p) = \frac{\partial y(t,p)}{\partial p}.$$

Determine a differential equation and initial condition for $v(t,p)$ that could be solved along with (1) to obtain $v(t,p)$. Assume as much smoothness as you like.

To avoid confusion or ambiguity, use the notation

$$\partial_i f(y(t,p),p)$$

to denote the derivative of f with respect to its i^{th} argument ($i \in \{1,2\}$), evaluated at $(y(t,p),p)$; and the notation

$$\frac{\partial}{\partial p} f(y(t,p),p)$$

to denote the derivative of the expression $f(y(t,p),p)$ with respect to p.

(b) Specialize your answer for part (a) to the case $f(y,t) = py^2 - p^3$, and write out the system of two coupled ODEs explicitly.

6. 2nd order ODE BVP: Galerkin method

[Answer this **OR** 5 **OR** 7.]

As you know, the Galerkin approximation $Y = \Sigma_{i=1}^{N} a_i \phi_i$ to the solution of the BVP $(-ry')' + sy = f$, $y(0) = y(1) = 0$, is defined by requiring the *residual* $(-rY')' + sY - f$ to be orthogonal to each of the $\{\phi_i\}$ with respect to the inner product $< g, h >_1 \equiv \int gh$, so that it minimizes the residual in the norm induced by this inner product.

But what about the actual *error* of the Galerkin approximation, $e = Y - y$ where y is the exact solution?

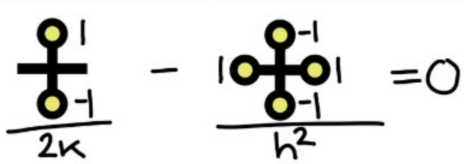
Show that e is orthogonal to each of the $\{\phi_i\}$ with respect to the inner product $< g, h >_2 \equiv \int \left(rg'h' + sgh \right)$ (if the functions r and s are such that this in fact qualifies as an inner product), so that the Galerkin approximation minimizes the error in the norm induced by this other inner product.

Hints: Replace f in the residual using the differential equation. Integrate by parts. Make use of the boundary conditions.

7. FD method stability and consistency

[Answer this **OR** 5 **OR** 6.]

The DuFort-Frankel finite difference scheme for the heat equation $u_t - u_{xx} = 0$ is



Determine any conditions on the grid spacings h (space) and k (time) that are necessary for the method to be *consistent* with the heat equation. To do this, Taylor-expand the Dufort-Frankel approximation to $u_t - u_{xx}$ and compare with $u_t - u_{xx}$ itself as $h \to 0$ and $k \to 0$.

8. Spectral differentiation

(a) Write down or compute the matrix, F_4, of the discrete Fourier transform (DFT) for $n = 4$.

(b) Find the DFT of the vector $x = [3, 1, 1, 3]^T$, representing the values of a function on the grid $\{t_j = \frac{j}{4}T \mid j = 0, 1, 2, 3\}$.

(c) Use your answer to part (b) to compute the spectral derivative of x.