(a) $y_{j+2} = y_j + 2h f_{j+1}$

ℓ	0	1	2
α_ℓ	-1		1
β_ℓ		2	

$$\sum_{\ell=0}^{K} \alpha_\ell y_{j+\ell} = h \sum_{\ell=0}^{K} \beta_\ell f_{j+\ell}.$$

$$c_0 = \sum_\ell \alpha_\ell = -1 + 0 + 1 = 0 \checkmark$$

$$c_1 = \sum_\ell \ell \alpha_\ell - \sum \beta_\ell = (0 \cdot -1 + 1 \cdot 0 + 2 \cdot 1) - (0 + 2 + 0)$$
$$= 2 - 2 = 0 \checkmark$$

$$c_2 = \frac{1}{2!}(0 + 2^2 \cdot 1) - \frac{1}{1!}(2 + 2^1 \cdot 0) = 2 - 2 = 0 \checkmark$$

$$c_3 = \frac{1}{3!}(0 + 2^3 \cdot 1) - \frac{1}{2!}(2 + 2^2 \cdot 0) = \frac{4}{3} - 1 = \frac{1}{3} \neq 0 \quad \times$$

Since $c_0 = c_1 = c_2 = 0$ and $c_3 \neq 0$,
the order of accuracy is $\underline{2}$.

An LMF is <u>consistent</u> if its order of accuracy ≥ 1.
Thus this method is consistent.

1 (b) stability?

$$\rho(z) = \sum_{\ell=0}^{2} \alpha_\ell z^\ell = -1 + 0 \cdot z + z^2$$
$$= z^2 - 1 = (z-1)(z+1)$$

Eigenvalues for $f \equiv 0$ are thus $z_\pm = \pm 1$.

Since $|z_\pm| \le 1$ and both are single roots,
ie. multiplicity 1
the method is stable.

(c) We have a theorem that
(consistent & stable) $\Leftrightarrow$ convergent.
Thus this method is convergent.

(d) For $f(t,y) = \lambda y$,

$$\sum_{\ell=0}^{2} \alpha_\ell y_{j+\ell} = h \sum_{\ell=0}^{2} \beta_\ell f_{j+\ell} = h\lambda \sum_{\ell=0}^{2} \beta_\ell y_{j+\ell}$$

$$\sum_{\ell=0}^{2} (\alpha_\ell - h\lambda\beta_\ell)\, y_{j+\ell} = 0$$

$$(-1-0)\, y_j + (0 - h\lambda \cdot 2)\, y_{j+1} + (1-0)\, y_{j+2} = 0$$

$$-y_j - 2h\lambda y_{j+1} + y_{j+2} = 0$$

$$z^2 - 2h\lambda z - 1 = 0$$

$$z = h\lambda \pm \sqrt{1 + (h\lambda)^2} \approx h\lambda \pm 1 \text{ for small } |h\lambda|.$$

For all small $h\lambda$, one of $|h\lambda \pm 1| > 1$.

Hence there is a growing mode even for small $h\lambda < 0$.

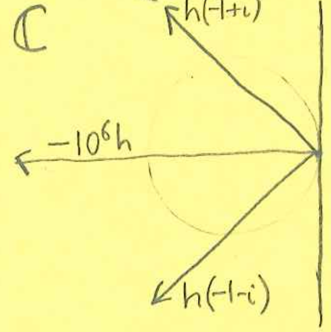
1(e). No. The blow-up is as $t \to \infty$.
For any fixed T, the method converges on $[t_0, T]$. as $h \to 0$, so it's usable.

2. (a) Any answer with a sensible justification accepted.

(b) The methods have the same order of accuracy, but RK requires 4 times as many function evaluations per step as AB.

(c) Because AB is a multistep formula built on uniform time steps, it is harder to incorporate into a method that adaptively adjusts the step size.
It is also more complicated to start up than RK.

(d) For a system with eigenvalues $-1 \pm i$ and -10^6 we want a stability region that contains the rays $h.(-1 \pm i)$, $-10^6 h$ for $h > 0$.



Backward Euler, which is A-stable, satisfies this requirement, and so do at least some others of the Backward Differentiation formulas.

None of the explicit methods will be practical because the -10^6 eigenvalue will require a tiny timestep for stability.

3 (a) Taylor series applied to $y' = \lambda y$, $\lambda \in \mathbb{C}$.

$$y' = \lambda y \rightarrow y'' = \lambda y' = \lambda(\lambda y) = \lambda^2 y \rightarrow \dots$$

$$y^{(k)} = \lambda^k y$$

For the n^{th} order method, then

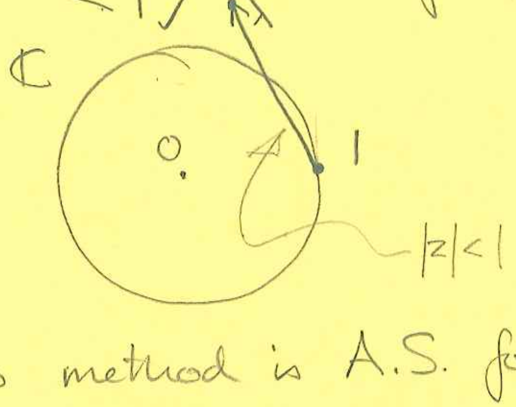
$$y_{j+1} = \sum_{k=0}^{n} \frac{h^k y^{(k)}(t_j)}{k!} = \underbrace{\left(\sum_{k=0}^{n} \frac{(h\lambda)^k}{k!}\right)}_{= z_n \text{, the eigenvalue.}} y_j$$

(b) Method is <u>absolutely stable</u> for step size h if $|z| < 1$.

$$|z_n| = \left| 1 + h\lambda + \frac{(h\lambda)^2}{2} + \dots + \frac{(h\lambda)^n}{n!} \right|$$

$$\overset{n \geq 1}{\approx} \left| 1 + h\lambda \right| \Big\} \text{ for all sufficiently small } h > 0 \text{ if } \mathrm{Re}(\lambda) < 0.$$

< 1

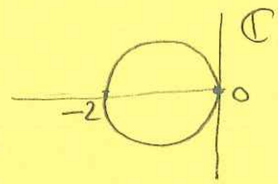


So method is A.S. for suff. small h if $\mathrm{Re}(\lambda) < 0$ and $n \geq 1$.

3

(c) $\underline{n=1}$

$$Z_1 = 1 + h\lambda = \pm 1 \Rightarrow h\lambda = 0, -2.$$



So lose stability at $h\lambda = -2$.

$\underline{n=2}$

$$Z_2 = 1 + h\lambda + \frac{(h\lambda)^2}{2} = \pm 1$$

$$\Rightarrow \begin{cases} h\lambda\left(1 + \frac{h\lambda}{2}\right) = 0 \Rightarrow h\lambda = 0, -2 \\ \text{or} \\ 2 + h\lambda + \frac{(h\lambda)^2}{2} = 0 \quad \text{which has no real roots.} \end{cases}$$

So again lose stability at $h\lambda = -2$.

(d) $\underline{n=3}\Big|_{h\lambda=-2}$

$$Z_3 = 1 + h\lambda + \frac{(h\lambda)^2}{2} + \frac{(h\lambda)^3}{3!}$$

$$Z_3\Big|_{h\lambda=-2} = 1 - 2 + \frac{(-2)^2}{2} + \frac{(-2)^3}{3!}$$

$$= -1 + 2 - \frac{8}{6}$$

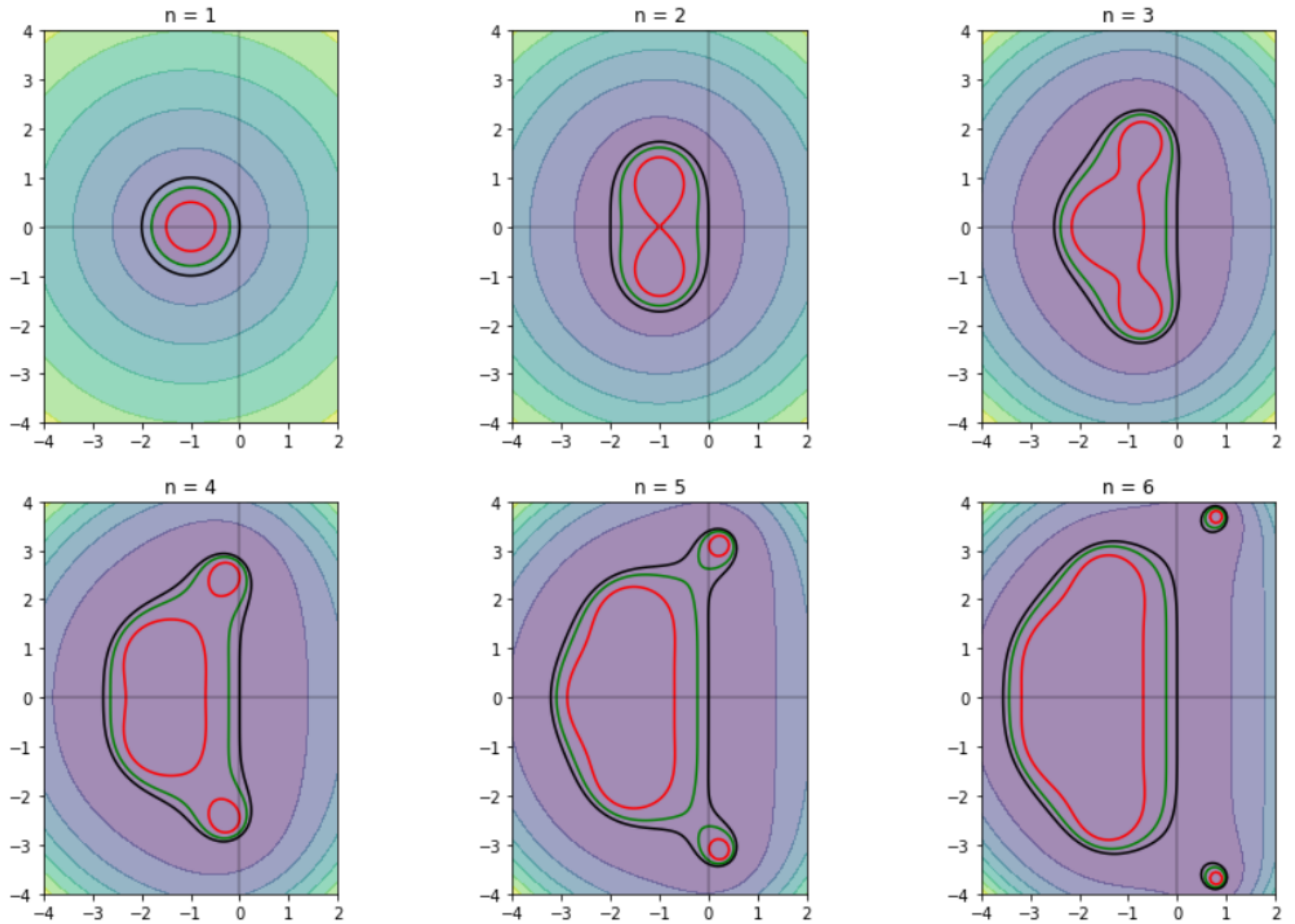
$$= 1 - \frac{4}{3} = -\frac{1}{3}$$

$\left|-\frac{1}{3}\right| < 1$, so $n=3$ method is still absolutely stable at $h\lambda = -2$.

Just out of curiosity, I plotted the stability regions,
which I don't recall having seen before:

```python
I = 1.j
fig,axs = plt.subplots(2,3,figsize=(15,10))
axs = axs.flatten()
iax = 0
for n in range(1,7):
    ax = axs[iax]
    iax += 1
    x = np.linspace(-4,2,100)
    y = np.linspace(-4,4,100)
    X,Y = np.meshgrid(x,y)
    s = [ ((X+I*Y)**i)/int(factorial(i)) for i in range(n+1)]
    z = np.sum( s, axis=0 )
    modz = np.abs(z)
    ax.set_aspect(1)
    ax.axhline(0,color='k',alpha=0.25)
    ax.axvline(0,color='k',alpha=0.25)
    ax.contourf(X,Y,modz,alpha=0.5)
    ax.contour(X,Y,modz,levels=[.5,.8,1],colors=['r','g','k'])  # black is |z|=1
    ax.set_title(f'n = {n}')
plt.suptitle('Stability boundary (black) for Taylor series methods') ;
```

Stability boundary (black) for Taylor series methods



4 RK. SOLUTIONS (8)

$$y_i^{[k+1]} = y_i^{[k]} + h\Big(b f_i(y^{[k]}) + c f_i\big(y^{[k]} + h a f(y^{[k]})\big)\Big)$$

$f_i(y+v) = f_i(y) + \partial_1 f_i(y) v_1 + \partial_2 f_i(y) v_2$

$$= y_i^{[k]} + h b f_i(y^{[k]})$$
$$+ hc\Big[f_i(y^{[k]}) + \partial_1 f_i(y^{[k]}) \cdot h a f_1(y^{[k]}) + \partial_2 f_i(y^{[k]}) \cdot h a f_2(y^{[k]})\Big]$$
$$+ O(h^2)$$

$$= y_i^{[k]} + h[b+c] f_i + h^2\big[ac(\partial_1 f_i \cdot f_1 + \partial_2 f_i \cdot f_2)\big] + O(h^3)$$

On the other hand,

$$y_i(t_k+h) = y_i(t_k) + h y_i'(t_k) + \frac{h^2}{2} y_i''(t_k) + O(h^3)$$

$y_i' = f_i(y)$

$$= y_i^{[k]} + h f_i + \frac{h^2}{2}(\partial_1 f_i \cdot f_1 + \partial_2 f_i \cdot f_2) + O(h^3)$$

(the last bracket annotated y_i'')

So we need

$$b + c = 1 \quad \text{and} \quad ac = \frac{1}{2}.$$

Setting $a = \frac{1}{2c}$, $b = 1-c$, we have a family of methods parametrized by c:

$$y^{[k+1]} = y^{[k]} + h\Big((1-c) f(y^{[k]}) + c f\big(y^{[k]} + \frac{h}{2c} f(y^{[k]})\big)\Big).$$

[Check w. Wikipedia

$c=1$: $y^{[k+1]} = y^{[k]} + h f\big(y^{[k]} + \frac{h}{2} f(y^{[k]})\big)$ ✓ "midpoint"

$c = \frac{1}{2}$: $y^{[k+1]} = y^{[k]} + \frac{h}{2}\Big(f(y^{[k]}) + f\big(y^{[k]} + h f(y^{[k]})\big)\Big)$ ✓ "Heun".]

(b) Any non-autonomous system of DEs can be autonomous at the cost of increasing the dimension by 1: make t a dependent variable, tau, governed by the DE dtau/dt = 1, with tau(t0) = t0.

5

(a) $v(t,p) = \frac{\partial y(t,p)}{\partial p}$

$$\frac{\partial v(t,p)}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\partial y(t,p)}{\partial p}\right) \overset{\text{Smooth}}{=} \frac{\partial}{\partial p}\left(\frac{\partial y(t,p)}{\partial t}\right)$$

$$= \frac{\partial}{\partial p}\left(f(y,p)\right)$$

$$= \partial_1 f(y(t,p),p)\frac{\partial y(t,p)}{\partial p} + \partial_2 f(y(t,p),p)\cdot 1$$

$$= \partial_1 f(y(t,p),p)\, v(t,p) + \partial_2 f(y(t,p),p)$$

Or, with evaluation points understood,

$$v' = \partial_1 f \cdot v + \partial_2 f \quad , \text{ and } v(0,p) = 0.$$

(b) For $f(y,p) = py^2 - p^3$

$$\partial_1 f = 2py \;, \quad \partial_2 f = y^2 - 3p^2$$

So we solve

$$y' = p \cdot y^2 - p^3 \;, \; y(0) = y_0$$

$$v' = 2pyv + y^2 - 3p^2 , \; v(0) = 0$$

6

y = true solution

Y = Galerkin approximation

Know $\int_0^1 \Big[\underbrace{(-rY')' + sY - f}_{\text{residual}}\Big]\phi_i = 0 \quad \forall i$

But $(-ry')' + sy = f$, so

$$\int_0^1 \Big[(-rY')' + sY - \big((-ry')' + sy\big)\Big]\phi_i = 0 \quad \forall i$$

or $$\int_0^1 \Big[\big(-r(Y-y)'\big)' + s(Y-y)\Big]\phi_i = 0 \quad \forall i$$

or $$\int_0^1 \Big[(-re')' + se\Big]\phi_i = 0 \quad \text{with } e = Y - y.$$

Integrating the first term by parts and noting $\phi_i(0) = \phi_i(1) = 0 \ \forall i$

$$\int (-re')'\phi_i = (-re')\phi_i\Big|_0^1 - \int (-re')\phi_i' = \int re'\phi_i'$$

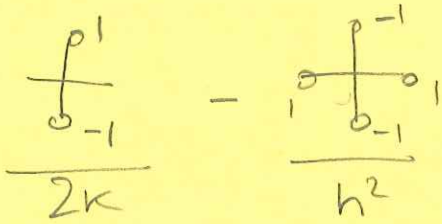
and so

$$\int_0^1 re'\phi_i' + se\phi_i = 0 \quad \forall i$$

i.e. e is orthogonal to the ϕ_i wrt the inner product $\langle u,v\rangle = \int ru'v' + suv$

if this is an inner product.

7



$$= \Big[u + k u_t + \frac{k^2}{2} u_{tt} + \frac{k^3}{6} u_{ttt} + \frac{k^4}{24} u_{tttt} + O(k^5) - \Big(u - k u_t + \frac{k^2}{2} u_{tt} - \frac{k^3}{6} u_{ttt} + \frac{k^4}{24} u_{tttt} \Big) \Big] \Big/ 2k$$

$$-\frac{1}{h^2} \Big[u - h u_x + \frac{h^2}{2} u_{xx} - \frac{h^3}{6} u_{xxx} + \frac{h^4}{24} u_{xxxx} + O(h^5) + u + h u_x + \frac{h^2}{2} u_{xx} + \frac{h^3}{6} u_{xxx} + \frac{h^4}{24} u_{xxxx}$$
$$- \Big(u + k u_t + \frac{k^2}{2} u_{tt} + \frac{k^3}{6} u_{ttt} + \frac{k^4}{24} u_{tttt} + O(k^5) + u - k u_t + \frac{k^2}{2} u_{tt} - \frac{k^3}{6} u_{ttt} + \frac{k^4}{24} u_{tttt} \Big) \Big]$$

$$= u_t - u_{xx} + \frac{k^2}{h^2} u_{tt} + \frac{k^2}{6} u_{ttt} - \frac{h^2}{12} u_{xxxx} + \frac{k^4}{h^2} u_{tttt}$$

$$+ (k^2/h^2)\, O(k^3) + O(k^4) + O(h^3)$$

$\longrightarrow u_t - u_{xx}$ if and only if $\frac{k}{h} \to 0$ as $h \to 0,\ k \to 0$.

SOLUTIONS (14)

8

(a) $$(F_4)_{ij} = \frac{1}{\sqrt{4}}\left(e^{-2\hat{i}\pi/4}\right)^{ij} = \left(e^{-\hat{i}\frac{\pi}{2}}\right)^{ij} = (-\hat{i})^{ij}$$

$(\hat{i}^2 = -1)$.

$$F_4 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ i & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix}.$$

(b) $$F_4\begin{bmatrix}3\\1\\1\\3\end{bmatrix} = \frac{1}{2}\begin{bmatrix} & & '' & & \end{bmatrix}\begin{pmatrix}3\\1\\1\\3\end{pmatrix} = \frac{1}{2}\begin{bmatrix}3+1+1+3\\3-i-1+3i\\3-1+1-3\\3+i-1-3i\end{bmatrix}$$

(slow Fourier transform!)

$$= \frac{1}{2}\begin{bmatrix}8\\2+2i\\0\\2-2i\end{bmatrix} = \begin{bmatrix}4\\1+i\\0\\1-i\end{bmatrix}$$

(c) $$\overbrace{\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}}^{F_4^{-1}}\left(\begin{bmatrix}4\\1+i\\0\\1-i\end{bmatrix} * \begin{array}{l} \begin{bmatrix}0\\1\\(0)\\-1\end{bmatrix} \end{array}\begin{array}{l} \leftarrow k=0 \\ \leftarrow k=1 \\ \leftarrow |k| = n/2 \\ \leftarrow k = -n/2+1 \end{array} \cdot \frac{2\pi\hat{i}}{T}\right)$$

$$= \frac{\pi i}{T}\begin{bmatrix} & '' & \end{bmatrix}\begin{bmatrix}0\\1+i\\0\\-1+i\end{bmatrix} = \frac{\pi i}{T}\begin{bmatrix}1+i+(-1+i)\\i(1+i)-i(-1+i)\\-(1+i)-(-1+i)\\-i(1+i)+i(-1+i)\end{bmatrix} = \frac{\pi i}{T}\begin{bmatrix}2i\\2i\\-2i\\-2i\end{bmatrix} = \frac{2\pi}{T}\begin{bmatrix}-1\\-1\\1\\1\end{bmatrix}$$