1 (a) $f\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) = \begin{bmatrix} -v \\ 9u \end{bmatrix}$ HW#1 SOLUTIONS

$$\left\| f\left(\begin{bmatrix} u_1 \\ v_1 \end{bmatrix}\right) - f\left(\begin{bmatrix} u_2 \\ v_2 \end{bmatrix}\right) \right\|_\infty = \left\| \begin{bmatrix} -v_1 \\ 9u_1 \end{bmatrix} - \begin{bmatrix} -v_2 \\ 9u_2 \end{bmatrix} \right\|_\infty$$

$$= \left\| \begin{matrix} v_2 - v_1 \\ 7(u_1 - u_2) \end{matrix} \right\|_\infty = \max\left(|v_2 - v_1|, 7|u_2 - u_1| \right)$$

On the other side,

$$\left\| \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} - \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} \right\|_\infty = \left\| \begin{matrix} u_1 - u_2 \\ v_1 - v_2 \end{matrix} \right\|_\infty = \max\left(|u_1 - u_2|, |v_1 - v_2| \right)$$

Thus

$$\begin{aligned} \left\| f\left(\begin{bmatrix} u_1 \\ v_1 \end{bmatrix}\right) - f\left(\begin{bmatrix} u_2 \\ v_2 \end{bmatrix}\right) \right\| &= \max\left(|v_2 - v_1|, 7|u_2 - u_1| \right) \\ &\leq \max\left(7|v_2 - v_1|\ 7|u_2 - u_1| \right) \\ &= 7 \max\left(|v_2 - v_1|, |u_2 - u_1| \right) \\ &= 7 \left\| \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} - \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} \right\|. \end{aligned}$$

Thus 7 is a Lipschitz constant for f. This applies to all $t \in \mathbb{R}$: since f does not actually depend on t in this case, I omitted the formal dependence on t. Any number greater than 7 also works.

1(b) We have this Theorem 7.2 from Ackleh et al (p 382):
Let f be a continuous vector-valued function defined on $S = \{(t,y) \mid t \in \mathbb{R}, y \in \mathbb{R}^n, |t-a| \leq \gamma, \|y\| < \infty\}$ satisfying a Lipschitz condition in y over S. Then $y' = f(t,y)$, $y(a) = y_0$ has a unique solution for $|t-a| \leq \gamma$.

In our IVP, S can be taken to be all of $\mathbb{R} \times \mathbb{R}^n$, i.e. γ can be taken as large as you like. So the answer is "yes", there is a unique solution extending over all $t \in \mathbb{R}$.

1(c) We are told $\left\{\begin{array}{l} u'(t) = -v(t) \\ v'(t) = 7u(t) \end{array}\right\}$ for all $t \in \mathbb{R}$.

Thus u', v' exist, i.e. u, v are differentiable.
Since a composition of differentiable functions is differentiable, $-v$ and $7u$ are differentiable.
Therefore, differentiating the DEs, we find

$$u''(t) = -v'(t) = -7u(t)$$
$$v''(t) = 7u'(t) = -7v(t)$$

Thus, Since $-7u$ and $-7v$ are differentiable, u'' and v'' are too.

Differentiating again, we find

$$u'''(t) = -7u'(t) = +7v(t)$$
$$v'''(t) = -7v'(t) = -49u(t),$$

So u''', v''' are also differentiable.
Continuing in the same way, we see u, v are infinitely many times differentiable, thus $\in C^\infty(\mathbb{R})$.

Note: The same conclusion can be drawn about the solution of any IVP $y' = f(t,y)$, $y(a) = y_0$ if the function f is C^∞.

2(a) Is $f(t,y) = y^2$ Lipschitz in y on $\mathbb{R} \times \mathbb{R}$?

The answer is no. I can see it graphically because

$\frac{\partial f}{\partial y}$ is unbounded as $|y| \to \infty$.

But formally suppose that it is. Then there exists a number L such that (w.o.l.o.g. $L > 0$)

$$\left| f(t, y_1) - f(t, y_2) \right| = \left| y_1^2 - y_2^2 \right| \leq L \left| y_1 - y_2 \right|$$

for all t, y_1, y_2.

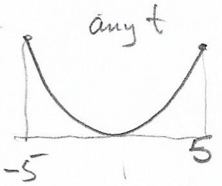
Take $y_1 = 2L$, $y_2 = L$. Then the above requirement is

$$\left| (2L)^2 - L^2 \right| = \left| 4L^2 - L^2 \right| = 3L^2 \leq L \left| 2L - L \right| = L^2,$$

which is false, contradicting the supposition.

2(b) Is f Lip in y on $\mathbb{R} \times [-5,5] \equiv S$?

$f(t,y) = y^2$



The answer is yes.

f is Lip on S if $\frac{\partial f_i}{\partial y_j}$ are continuous and bounded on S, all $i,j \in \{1,2,\dots,n\}$. Then

$$f_i(t,y) - f_i(t,\bar{y}) = \sum_{j=1}^{n} \frac{\partial f_i}{\partial y_j}(t,c_i)\cdot(y_j - \bar{y}_j)$$

for some c_i on the segment between y & $\bar{y}$. See Ackleh p383

Then

$$\|f(t,y) - f(t,\bar{y})\|_\infty$$

$$= \max_i |f_i(t,y) - f_i(t,\bar{y})|$$

$$= \max_i \left| \sum_{j=1}^{n} \frac{\partial f_i}{\partial y_j}(t,c_i)\cdot(y_j - \bar{y}_j)\right|$$

$$\leq \max_i \left| \sum_{j=1}^{n} \max_S \left|\frac{\partial f_i}{\partial y_j}\right| \cdot (y_j - \bar{y}_j)\right|$$

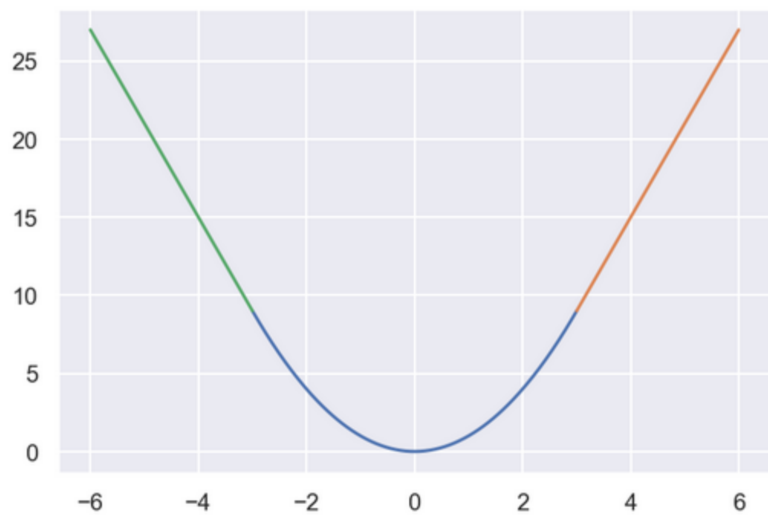
$$\leq \max_i \max_S \left|\frac{\partial f_i}{\partial y_j}\right| \cdot \sum_j |y_j - \bar{y}_j|$$

$$\leq \underbrace{\max_i \max_S \left|\frac{\partial f_i}{\partial y_j}\right|}\, n\|y - \bar{y}\|_\infty$$

This then is a Lip. const, L.

In our case, $n=1$, $\frac{\partial f}{\partial y} = 2y$. $L = \max_{\mathbb{R}\times[-3,3]} |2y| = 6$.

2(c)



This is continuous, exactly y^2 on $[-3, 3]$, but linear outside this interval.

It is also C^1 everywhere including at $y = \pm 3$. We could make it even smoother if we wished.

3ⓐ Lorenz system.

Local existence/uniqueness?

Yes, because f is polynomial in $Y = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

and independent of t,

and so is continuous with continuous partial derivatives on some open box containing (t, Y_0).

The IVP therefore satisfies the hypotheses of Ackleh Thm 7.1, and a unique solution is guaranteed for some open t-interval containing 0.

3(b) No. The partial derivatives of f are

$$\left\{ \frac{\partial f_i}{\partial y_j} \right\} = \begin{bmatrix} -10 & 10 & 0 \\ 28-z & -1 & -x \\ y & x & -\frac{8}{5} \end{bmatrix}$$

Some of these are unbounded on $\mathbb{R} \times \mathbb{R}^3$.

Therefore f is not Lipschitz in $Y = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

Another approach is to observe that

$$f\left(\begin{bmatrix} x \\ x \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 27x \\ x^2 \end{bmatrix}$$

So for sufficiently large x

$$\left\| f\begin{bmatrix} x \\ x \\ 0 \end{bmatrix} - f\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\|_\infty = x^2 > L \left\| \begin{bmatrix} x \\ x \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\| = Lx.$$

for any L.

3(c) Solution for all $t \in \mathbb{R}$.

From Ackleh Thm 7.2 we are not able to say, because f is not Lipschitz on $\mathbb{R} \times \mathbb{R}^3$, as we said in part (b).

3(d) Yes. The components of f are polynomials, so their partial derivatives are continuous and bounded on this closed bounded box in $\mathbb{R}^3$.

So f is Lipschitz on $\mathbb{R} \times \underbrace{[-30,30] \times [-30,30] \times [0,50]}_{\equiv B}$.

3(e) If I only care what happens in box B, then I can replace f by $\tilde{f}$, continuous w continuous partial derivatives

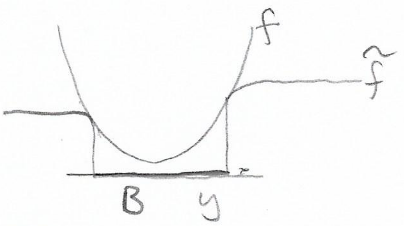
$$\tilde{f} = \begin{cases} f & \text{on } B \\ \text{something else that is Lipschitz on } \mathbb{R} \times \mathbb{R}^3 & \text{outside of } B. \end{cases}$$

$Y' = \tilde{f}$ has solutions identical to those of $Y' = f$ as long as $Y(t)$ remains in B.

$Y' = \tilde{f}$, $Y(0) = Y_0 \in B$ has a unique solution on $t \in \mathbb{R}$.

Therefore $Y' = f$, $Y(0) = Y_0 \in B$ has a unique solution at least until $Y(t)$ first exits B.

Example:



1(f) Lorenz with Euler - want to see evidence of convergence

```
from matplotlib import rcdefaults
rcdefaults()  # restore default matplotlib rc parameters
%config InlineBackend.figure_format='retina'
import seaborn as sns  # wrapper for matplotlib that provides prettier styles and more
import matplotlib.pyplot as plt  # use matplotlib functionality directly
%matplotlib inline
sns.set()
```

```
import numpy as np
```

```
def numerical_solution(m,Y0,T,method,color='k',plot_type='time'):

    global nexty,f,ax

    t = 0
    y = np.array(Y0)

    h = (T-t)/m

    ta = np.empty(m+1)
    ta[0] = t

    ya = np.empty((len(Y0),m+1))  # a 2d array
    ya[:,0] = y
    for k in range(m): # 0, 1, 2, ...
        if method=='euler':
            y = y + h*f(t,y)  # scalar mult and addition done element-wise
        else:
            assert(False) # exit if unimplemented method requested
        t += h
        ta[k+1] = t
        ya[:,k+1] = y

    if plot_type == 'time':
        d = len(Y0)
        for i in range(d):
            if i==0:
                ax.plot(ta,ya[i,:],'-',color=color,label=str(m)) # plot ith component vs t
            else:
                ax.plot(ta,ya[i,:],'-',color=color) # plot ith component vs t
        ax.legend(loc='lower center')
    elif type(plot_type) == list:
            i,j = plot_type
            ax.plot(ya[i,:],ya[j,:],'-',color=color) # plot ith component vs jth
    return t,y  # return the final point


```

1(f) continued

Generate the numerical approximate solutions

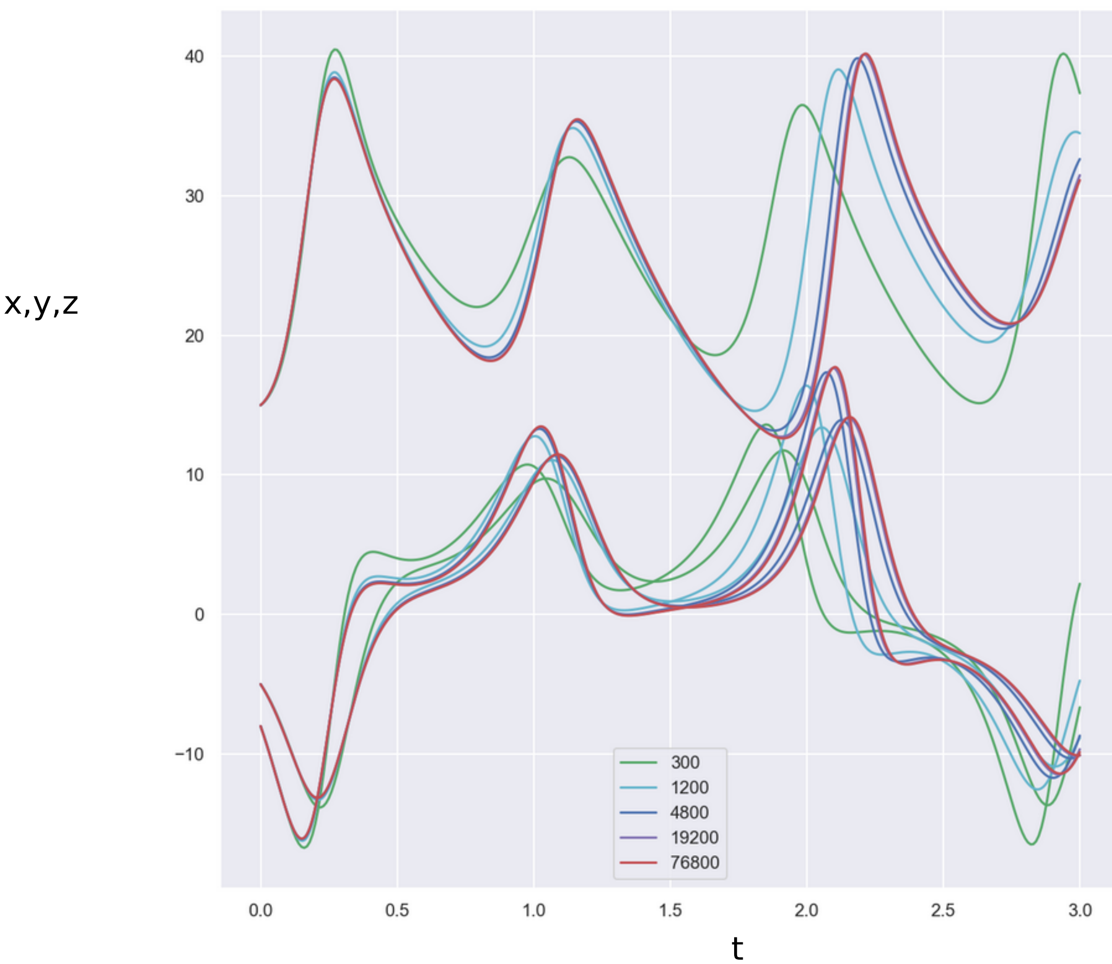
```
def f(t,Y):
    x,y,z = Y
    return np.array([ 10*y - 10*x, 28*x -y -x*z, -8*z/5 + x*y ])

Y0 = np.array([-5.,-8.,15.])   # changed from -4,-8,15 for s22
T = 1.
doplot = True

fig = plt.figure(figsize=(10,10))
ax = plt.subplot(111)

colors = 'gcbmr'
nc = 0
for method in ['euler']:
    for m in [300*4**i for i in range(5)]:
        print(m)
        t1,y1 = numerical_solution(m,Y0,3.0,method,color=colors[nc])
        nc += 1
        print(y1)  # to see final value of y
```

```
300
[-6.65094034  2.20345286 37.3528184 ]
1200
[-8.82800314 -4.74038825 34.48436814]
4800
[-10.082517    -8.68367443  32.63858672]
19200
[-10.11171328  -9.65544671  31.48003229]
76800
[-10.08632628  -9.87400475  31.12889622]
```



We can see some evidence for convergence with as the number of steps gets into the 100,000 range.

1(f) continued

Now the coordinate plane projections:

```
fig,axs = plt.subplots(1,3,figsize=(15,5))
nc = 0
for method in ['euler']:
    for m in [300*4**i for i in range(5)]:
        print(m)
        vars = [0,1]; ax = axs[0]
        t1,y1 = numerical_solution(m,Y0,3.0,method,color=colors[nc],plot_type=vars)
        ax.set_xlabel('x'); ax.set_ylabel('y')

        vars = [0,2]; ax = axs[1]
        t1,y1 = numerical_solution(m,Y0,3.0,method,color=colors[nc],plot_type=vars)
        ax.set_xlabel('x'); ax.set_ylabel('z')

        vars = [1,2]; ax = axs[2]
        t1,y1 = numerical_solution(m,Y0,3.0,method,color=colors[nc],plot_type=vars)
        ax.set_xlabel('y'); ax.set_ylabel('z')

        nc += 1
        print(y1)  # to see final value of y
```

```
300
[-6.65094034  2.20345286 37.3528184 ]
1200
[-8.82800314 -4.74038825 34.48436814]
4800
[-10.082517    -8.68367443  32.63858672]
19200
[-10.11171328  -9.65544671  31.48003229]
76800
[-10.08632628  -9.87400475  31.12889622]
```