Homework 5

due 11:59pm Thursday, March 31

Complete illustrated "stories" to Gradescope. Jupyter notebook with code to UBLearns.

1. BVP by finite differences

Solve the following BVP by the method of finite differences:

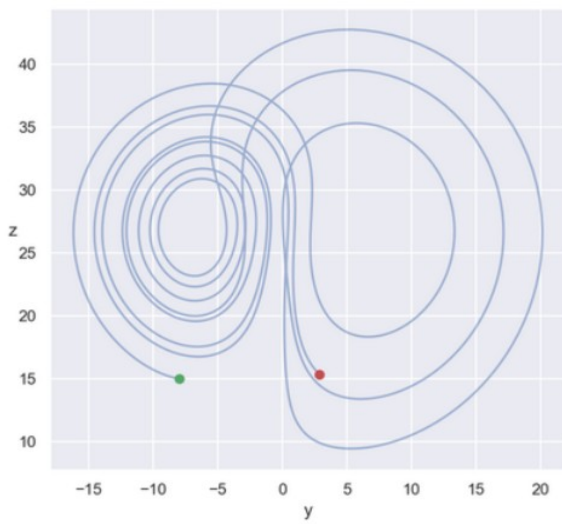
$$y'' + 20\sin(20x)y' - \frac{6}{x^2}y = 1$$

for $x \in (2, 4)$, and $y(2) = 1$, $y(4) = 8$.

Print your (best) estimate for y(3), and provide some empirical evidence that the maximum error is $O(h^2)$.

2. Variational equations in Lorenz system

(a) Solve the variational equations for the Lorenz system IVP given in Homework #3 Q1 using the (fixed) RK4 code.



Hint: Numpy functions `reshape` and `append` will be useful for packing and unpacking Y and V into/from a single 1D array of length $n + n^2$.

Print the values of x, y, z at $t = 10$.

Calculate the matrix V(10), and its *singular values* (https://en.wikipedia.org/wiki/Singular_value_decomposition, np.linalg.svd).

(b) Interpret each of the singular values. One of them reflects the "butterfly effect" present in this system: sensitive dependence on initial conditions.

(Parenthetical note: An accurate computation of these singular values would involve re-orthogonalizing the matrix V(t) periodically to avoid collapse of its columns along the most rapidly expanding direction.)

3. Dependence of the solution of an ODE IVP on a parameter

In question 2, you computed the sensitivity of a solution of an ODE IVP to the initial condition by co-solving another (coupled) IVP (the "variational equations"). Now consider the initial value y_0 to be fixed, but the differential equation depends on a parameter, p:

$$\frac{\partial y(t,p)}{\partial t} = f(y,p), \qquad y(0,p) = y_0 \qquad (1)$$

For simplicity we will consider an autonomous scalar equation depending on a scalar parameter. We want to know the derivative of the solution at time t with respect to the parameter p.

(a) Call the derivative we are interested in knowing $v(t,p)$:

$$v(t,p) = \frac{\partial y(t,p)}{\partial p}.$$

Determine a differential equation and initial condition for $v(t,p)$ that could be solved along with (1) to obtain $v(t,p)$. Assume as much smoothness as you like.

To avoid confusion or ambiguity, use the notation $\partial_i f(y(t,p),p)$ to denote the derivative of f with respect to its i^{th} argument ($i \in \{1,2\}$), evaluated at $(y(t,p),p)$; and the notation

$$\frac{\partial}{\partial p} f(y(t,p),p)$$

for the derivative of the expression $f(y(t,p),p)$ with respect to p.

(b) Specialize your answer for part (a) to the case $f(y,p) = py^3$, and write it the way we would if we were about to code it up for numerical solution.