5

DuFort-Frankel heat equ stability SOLUTION

$$\frac{1}{2k}\begin{array}{c} 1 \\ \text{—}\!\!\!\!\!\!\;| \\ -1 \end{array} = \frac{1}{h^2}\begin{array}{ccc} & -1 & \\ 1 & \text{—}\!\!\!\!\!\!\;| & 1 \\ & -1 & \end{array}$$

Try $u_{i,j} = e^{I\alpha i}\lambda^j$; Set $\sigma = k/h^2$.

$$\frac{1}{2}(\lambda - \lambda^{-1}) = \sigma(e^{-I\alpha} + e^{I\alpha} - \lambda - \lambda^{-1})$$

$$\lambda^2 - 1 = 2\sigma(\lambda 2\cos\alpha - \lambda^2 - 1)$$

$$\boxed{(1+2\sigma)\lambda^2 - 4\sigma\cos\alpha\,\lambda + (2\sigma - 1) = 0} \quad \text{Char. poly.}$$

$$\lambda_\pm = \frac{2\sigma\cos\alpha \pm \sqrt{4\sigma^2\cos^2\alpha - (1+2\sigma)(2\sigma-1)}}{(1+2\sigma)}$$

$$= \frac{2\sigma\cos\alpha \pm \sqrt{1 - 4\sigma^2\sin^2\alpha}}{(1+2\sigma)}$$

Case $\alpha = 0$: $\lambda_\pm = \frac{2\sigma \pm 1}{2\sigma + 1} = 1, \frac{2\sigma - 1}{2\sigma+1} \in (-1,1)$ for $\sigma > 0$.

Case $\alpha = \pi$: $\lambda_\pm = \frac{-2\sigma \pm 1}{2\sigma+1} = -1, \frac{-(2\sigma-1)}{2\sigma+1} \in (-1,1)$ for $\sigma > 0$.

Case $\lambda_\pm$ complex: $|\lambda_\pm|^2 = \lambda_+\lambda_- = \frac{4\sigma^2\cos^2\alpha - (1 - 4\sigma^2\sin^2\alpha)}{(1+2\sigma)^2}$

$$= \frac{4\sigma^2 - 1}{(2\sigma+1)^2} = \frac{2\sigma - 1}{2\sigma+1} \in (-1,1) \text{ for } \sigma > 0$$

Case $\lambda_\pm$ real, $\alpha \in (0,\pi)$: seek $\lambda = \pm 1$.

$\lambda = \pm 1$: $(1+2\sigma) \mp 4\sigma\cos\alpha + 2\sigma - 1 = 0$

$$4\sigma(1 \mp \cos\alpha) = 0$$

$\sigma > 0 \Rightarrow \cos\alpha = \pm 1$, contradicting $\alpha \in (0,\pi)$.

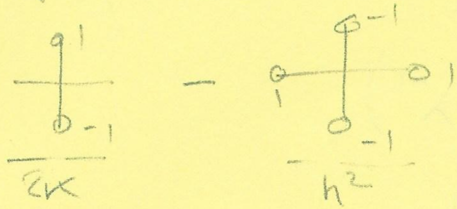
Thus no exit from $|\lambda| < 1$.

Thus scheme is stable for all $\sigma > 0$.

Final Exam Solutions

DuFort-Frankel consistency (heat eqn.)

$\mathcal{L} = u_t - u_{xx}$ [solution of kind expected]

$L =$ 

$$= \frac{u(t+k,x) - u(t-k,x)}{2k} - \frac{u(t,x-h) + u(t,x+h) - u(t+k,x) - u(t-k,x)}{h^2}$$

$$= \frac{\left[\begin{array}{l} u + ku_t + \frac{k^2}{2}u_{tt} + \frac{k^3}{6}u_{ttt} + \frac{k^4}{24}u_{tttt} + O(k^5) \\ -\left(u - ku_t + \frac{k^2}{2}u_{tt} - \frac{k^3}{6}u_{ttt} + \frac{k^4}{24}u_{tttt} + O(k^5)\right) \end{array}\right]}{2k}$$

$$- \frac{\left[\begin{array}{l} + u - hu_x + \frac{h^2}{2}u_{xx} - \frac{h^3}{6}u_{xxx} + \frac{h^4}{24}u_{xxxx} + O(h^5) \\ + u + hu_x + \frac{h^2}{2}u_{xx} + \frac{h^3}{6}u_{xxx} + \frac{h^4}{24}u_{xxxx} + O(h^5) \\ -\left(u + ku_t + \frac{k^2}{2}u_{tt} + \frac{k^3}{6}u_{ttt} + \frac{k^4}{24}u_{tttt} + O(k^5)\right. \\ -\left(u - ku_t + \frac{k^2}{2}u_{tt} - \frac{k^3}{6}u_{ttt} + \frac{k^4}{24}u_{tttt} + O(k^5)\right. \end{array}\right]}{h^2}$$

$$= u_t + \frac{k^2}{6}u_{ttt} + O(k^4)$$

$$- u_{xx} - \frac{k^2}{h^2}u_{tt} + \frac{h^2}{12}u_{xxxx} + \frac{k^4}{12h^2}u_{tttt} + O(k^5)/h^2 + O(h^3)$$

$\uparrow$

Not consistent with heat equation unless k goes to 0 faster than h.

For some reason, I had it in mind that I'd asked you to use SOR for this. But I didn't, and everyone used a direct solve, so there is no issue of convergence wrt iterations of SOR, only with respect to grid resolution.

2.

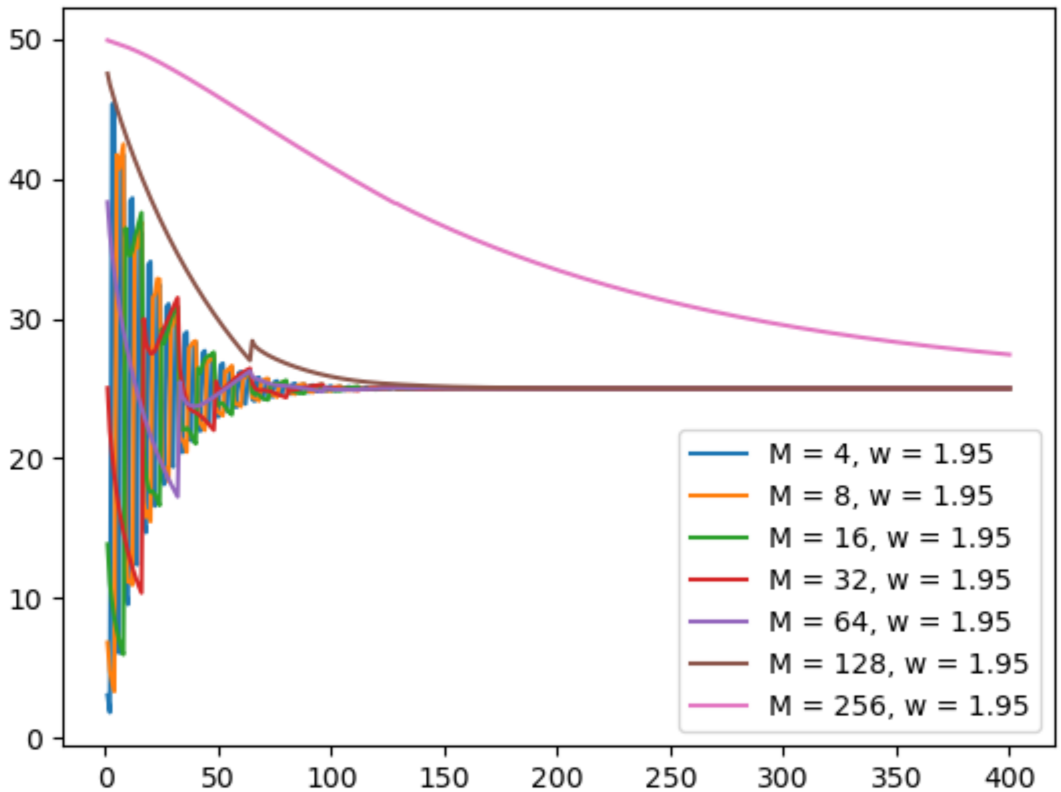
Use your Poisson code to determine the temperature at the center of the square plate in the BVP we discussed in class. Show some evidence of convergence of this estimate as the grid is refined.

If your length and width were the same, you should converge unusually rapidly to 25 because of symmetry.

I modified the SOR code I shared with you to perform the computation at multiple resolutions:

```
for M in [2**i for i in range(2,9)]:
```

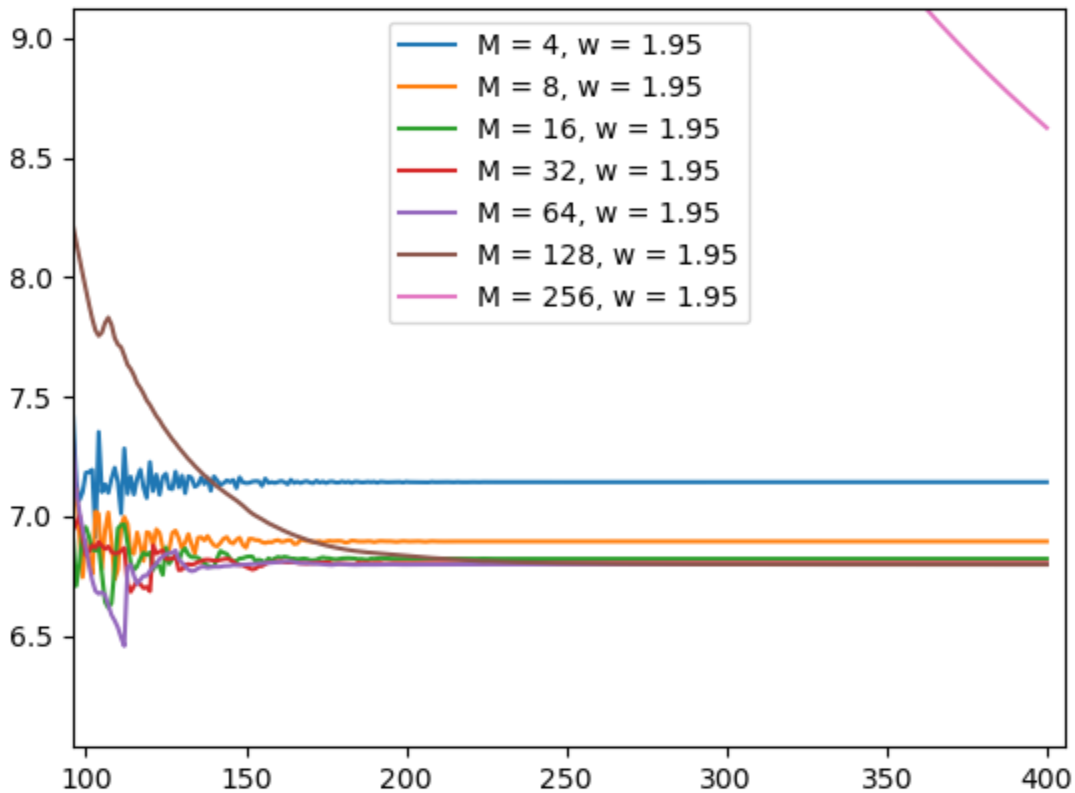
Here are plots of the value of the approximation at the center of the plate on the vertical axis vs iteration number on the horizontal axis.



I do not see any covergence of the value to which the SOR iteration is converging with respect to grid refinement. All I see is that the convergence becomes slow for M = 128 and very slow for M = 256. The latter might be improved if we tried a larger value of w for these fine grids.

I suspect that the lack of convergence with respect to resolution is because the finite difference scheme at any resolution gives the exact solution to the PDE at the center due to symmetry. So this was a bit of a trick question.
We can check this hypothesis by looking at a point not in the very center ...

Here are the plots for the point [M//4,N//4]:



Now we can see that the value to which the SOR iteration converges now shows convergence with respect to grid refinement, except that for the finest grid, M = 256, the SOR iteration is not even close to converging after 400 iterations.