538 Exam 2

Tuesday, April 15, 2025

1 Shooting

a

Convert the following 2nd order BVP to a first order one:

$$y'' = y' - 2y \quad \text{for} \quad x \in (5, 7), \qquad y(5) = 3, \quad y'(7) = 0.$$

Note that the boundary conditions are on the **value** of y at the left and on the **derivative** on the right.

b

With any two shots you like from $x = 5$, each using just **a single Euler step** to go all the way, find an approximation the value of $y(7)$. Note that $h = 7 - 5 = 2$. Do not find or use the exact solution of the differential equation.

2 Two of three other BVP methods

Obtain approximate solutions of the following boundary value problem using any **2 of the 3 methods specified below,**

$$\left((1+x)y'\right)' + y = 7x \quad \text{for} \quad x \in (0, 1), \qquad y(0) = y(1) = 0.$$

In each case, give the approximation for $y(\frac{1}{2})$.

(i) by finite differences, with nodes at $x = 0, \frac{1}{2}, 1$;

(ii) by collocation at $x = \frac{1}{2}$ with a basis of a single function $\phi(x) = x(1-x)$;

(iii) by the Galerkin method with the same single basis function $\phi(x) = x(1-x)$.

For method (iii), expressing the optimal coefficient as the ratio of integrals of two explicit polynomials will gain almost full credit. Only if you have time at the end, obtain the coefficient numerically.

3 Shooting for a non-autonomous linear BVP

(Extra credit)

Consider a system of differential equations $y : \mathbb{R} \to \mathbb{R}^n$ of the form

$$y'(t) = A(t)y(t) + f(t)$$

where A is a continuous matrix-valued function and f is a continuous vector-valued function.

Let $y_1(t)$ and and $y_2(t)$ be the two solutions with initial values $y_1(0)$ and $y_2(0)$ respectively. Solving the BVP using just two shots relies on the difference $y_1(t) - y_2(t)$ being a linear function of the initial difference $y_1(0) - y_2(0)$, that is $y_1(t) - y_2(t) = M(t)\,(y_1(0) - y_2(0))$ for some t-dependent matrix $M(t)$.

What is $M(t)$? Hint: try the scalar problem ($n = 1$) first.

SHOOTING $\quad y'' = y' - 2y \quad x \in (5,7)$

$y(5) = 3 \ , \ y'(7) = 0$

1st order system form

$$y' = v \qquad y(5) = 3$$
$$v' = v - 2y \qquad v(7) = 0$$

Euler with h = 2

$$\begin{bmatrix} y_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} y_0 \\ v_0 \end{bmatrix} + 2 \begin{bmatrix} v_0 \\ v_0 - 2y_0 \end{bmatrix}$$

First shot : $y_0 = 3 \ , \ v_0 = 0$ ← my choice

$$\begin{bmatrix} y_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 0 - 6 \end{bmatrix} = \begin{bmatrix} 3 \\ -12 \end{bmatrix}$$

Second shot $y_0 = 3 \ , \ v_0 = 1$ ← my 2nd choice

$$\begin{bmatrix} y_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 1 - 6 \end{bmatrix} = \begin{bmatrix} 3 + 2 \\ 1 - 10 \end{bmatrix} = \begin{bmatrix} 5 \\ -9 \end{bmatrix}$$

Summarizing :

v_0	v_1	
0	-12	+3
1	-9	
4	0	= prescribed value.

Extrapolating 4

Solution shot :

$$\begin{bmatrix} y_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} + 2 \begin{bmatrix} 4 \\ 4 - 6 \end{bmatrix} = \begin{bmatrix} 3 + 8 \\ 4 - 4 \end{bmatrix} = \begin{bmatrix} 11 \\ 0 \end{bmatrix}$$

Answer : $\boxed{y(7) \approx 11}$. with $y'(5) = 4$.

FINITE DIFFERENCES

$$-(r y')' + sy = f \qquad ((1+x)y')' + y = 7x$$

$$\underset{\underset{p}{\parallel}}{-r}y'' \; \underset{\underset{q}{\parallel}}{-r'}y' + sy = f$$

$$y''(\tfrac{1}{2}) \approx \frac{0 - 2w_1 + 0}{(\frac{1}{2})^2} = -8w_1$$

$$y'(\tfrac{1}{2}) \approx \frac{0 - 0}{2h} = 0$$

$$-8w_1 p(\tfrac{1}{2}) + 0 + w_1 s(\tfrac{1}{2}) = f(\tfrac{1}{2})$$

$$w_1 = \frac{f(\frac{1}{2})}{s(\frac{1}{2}) - 8p(\frac{1}{2})}$$

$$\left.\begin{aligned} p(x) &= +(1+x), & p(\tfrac{1}{2}) &= +\tfrac{3}{2} \\ q(x) &= 1, & q(\tfrac{1}{2}) &= 1 \\ s(x) &= 1, & s(\tfrac{1}{2}) &= 1 \\ f(x) &= 7x & f(\tfrac{1}{2}) &= \tfrac{7}{2} \end{aligned}\right\} \quad w_1 = \frac{\frac{7}{2}}{\underbrace{1 - 8\cdot(\frac{3}{2})}_{=-11}} = \boxed{\frac{-7}{22}} = -.3\overline{18}$$

COLLOCATION $\quad ((1+x)y')' + y = 7x$

$$(1+x)y'' + y' + y = 7x$$

$$y = ax(1-x) = ax - ax^2 \quad , \quad y(\tfrac{1}{2}) = a(\tfrac{1}{2}-\tfrac{1}{4}) = \tfrac{1}{4}a$$

$$y' = a - 2ax \quad , \quad y'(\tfrac{1}{2}) = a(1-2\cdot\tfrac{1}{2}) = 0$$

$$y'' = -2a \quad \quad y''(\tfrac{1}{2}) = -2a$$

At $x = \frac{1}{2}$, need

$$(1+\tfrac{1}{2})(-2a) + 0 + \tfrac{1}{4}a = 7\cdot\tfrac{1}{2}$$

$$-\frac{6}{2}a + \frac{1}{4}a = \frac{7}{2}$$

$$\frac{-12+1}{4}a = \frac{7}{2}$$

$$a = -\frac{14}{11} \quad , \quad \boxed{\tilde{y} = -\frac{14}{11}x(1-x)}.$$

$$w_1 = a\phi(\tfrac{1}{2}) = -\frac{14}{11}\cdot(\tfrac{1}{2})(1-\tfrac{1}{2}) = \boxed{-\frac{7}{22}} = -.3\overline{18}$$

(Same as FD).

GALERKIN

$$((1+x)y')' + y = 7x$$

$$-(\underbrace{-(1+x)}_{r(x)}\, y') + \underbrace{y}_{s(x)} = \underbrace{7x}_{f(x)}$$

$$\phi(x) = x(1-x) \qquad \tilde{y} = a\phi$$

$$\phi'(x) = 1-2x \qquad \tilde{y}' = a\phi'$$

Galerkin requires $\int_0^1 r(a\phi')\phi' + s(a\phi)\phi = \int_0^1 f\phi$

$$a\left[\int_0^1 -(1+x)\underbrace{(1-2x)^2}_{4x^2-4x+1} + 1\cdot(x(1-x))^2\,dx\right] = \int_0^1 7x\cdot x(1-x)\,dx$$

$$a\left[\int_0^1 \underbrace{-4x^3+3x-1 + x^4-2x^3+x^2}_{x^4-6x^3+x^2+3x-1}\,dx\right] = \int -7x^3+7x^2\,dx$$

$$a\left[\frac{x^5}{5} - \frac{6}{4}x^4 + \frac{x^3}{3} + \frac{3x^2}{2} - x\right]_0^1 = \left[-\frac{7}{4}x^4 + \frac{7}{3}x^3\right]_0^1$$

$$a\left[\frac{1}{5} - \frac{6}{4} + \frac{1}{3} + \frac{3}{2} - 1\right] = \left[-\frac{7}{4} + \frac{7}{3}\right]$$

$$a\left[\frac{-28}{60}\right] = \frac{7}{12}, \qquad a = -\frac{7}{12}\cdot\frac{15}{7} = -\frac{15}{12} = -\frac{5}{4}$$

$$\boxed{\begin{aligned}\tilde{y}(x) &= -\frac{5}{4}x(1-x)\\ \tilde{y}\left(\tfrac{1}{2}\right) &= -\frac{5}{16}\end{aligned}} = -.3125$$

Basis of 2-shot shooting for linear DE.

In exam question $y = $ "y_1"

$$y'(t) = A(t)y(t) + f(t).$$

Take another solution w. different initial value.

$\hat{y} = $ "y_2"

$$\hat{y}'(t) = A(t)\hat{y}(t) + f(t)$$

Then

$$\big(\underbrace{\hat{y}(t) - y(t)}_{\text{call this } v(t)}\big)' = A(t)\big(\hat{y}(t) - y(t)\big)$$

($f(t)$'s cancel)

$$\boxed{v'(t) = A(t)\,v(t)}.$$

In scalar case

$$\int_0^t \frac{v'}{v}\,dt = \int_0^t A\,dt$$

$$\ln|v(t)| = \int A\,dt$$

$$|v(t)| = v(0)e^{\int A(t)dt}$$

Likewise for the system with $n > 1$,

$$v(t) = v(0)e^{\int A(t)dt} \quad \text{(matrix exponential)}$$

Thus $\hat{y}(t) - y(t) = M(t)\big(\hat{y}(0) - y(0)\big)$

where $M(t) = e^{\int A(t)dt}$.