Homework #1

Due at 11:59pm, Friday, Feb 7, 2025.

In your answers to questions 1 and 3, refer to appropriate theorems of Ackley at al., Ch. 7.

1

Consider the following Initial Value Problem.

$$Y' = \begin{bmatrix} u' \\ v' \end{bmatrix} = f(t, Y) = \begin{bmatrix} -v \\ 5u \end{bmatrix}, \qquad Y(0) = \begin{bmatrix} u(0) \\ v(0) \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \end{bmatrix}.$$

(a) Is f Lipschitz in Y on $\mathbb{R} \times \mathbb{R}^2$? If yes, give a Lipschitz constant, L, with respect to a norm of your choice, stating which norm you are using. If not, explain why not.

(b) Does the IVP have a (unique) solution for all $t \in \mathbb{R}$? Explain how you know, or why it's not possible to say.

(c) How smooth is the solution of this IVP? That is, how many continuous derivatives (with respect to t) does it have? You can gain information iteratively by starting with the differential equations and (repeatedly) differentiating them (w.r.t. t). Recall that a differentiable function is necessarily continuous.

2

(a) Is the function $f(t, y) = y^2$ Lipschitz in y on $\mathbb{R} \times \mathbb{R}$?

(b) Is it Lipschitz in y on $\mathbb{R} \times [-5, 5]$? If so, provide a Lipschitz constant, L. Use the absolute value for the norm on $\mathbb{R}$.

(c) If it's possible, specify and sketch the graph of a function $g(y)$ which has the value y^2 on $[-5, 5]$, but is Lipschitz on all of $\mathbb{R}$.

3

Consider the following initial value problem (Lorenz).

$$Y' = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = f(t, Y) = \begin{bmatrix} 10y - 10x \\ 28x - y - xz \\ -\frac{8}{5}z + xy \end{bmatrix}, \qquad Y(0) = \begin{bmatrix} x(0) \\ y(0) \\ z(0) \end{bmatrix} = \begin{bmatrix} -4 \\ -8 \\ 14 \end{bmatrix}.$$

(a) Does it have a unique solution on some open t-interval containing 0? State either how do you know, or why it's not possible to say.

(b) Is f Lipschitz in Y on $\mathbb{R} \times \mathbb{R}^3$? Explain.

(c) Does the IVP have a unique solution for all $t \in \mathbb{R}$? State either how you know, or why it's not possible to say.

(d) Is f Lipschitz in Y on $\mathbb{R} \times ([-50, 50] \times [-50, 50] \times [0, 100])$? Explain.

(e) Suppose you only care about the solution as long as it remains in the box $Y \in [-50, 50] \times [-50, 50] \times [0, 100]$. What can you say about the existence and uniqueness of a solution until such time that it exits the box? Hint: consider another system that agrees exactly with this one in the box, but can differ outside it.

(f) Use your implementation of Euler's method in Python to approximate a solution (if it exists) of the IVP for $t \in [0, 4]$. Choose several values of the number of steps to illustrate (suggest, not prove) the convergence of the approximation as $h \to 0$. You might need quite a lot of steps. Make plots of x, y, z versus t, and also of the three projections y versus x and z versus x and z versus y. Try to label the plots so viewers can easily understand what they are seeing.