538 Spring 2025
Homework 3 Solutions

1. Eigenvalues of Adams-Bashforth methods

(a) Compute or look up the coefficients (alphas and betas) of the 2-step Adams-Bashforth method.

(b) Write down the characteristic polynomial of the method in the variable z when applied to the scalar equation $y' = \lambda y$ with step size h, and compute the eigenvalues.

(c) Show that one of the eigenvalues approximates the one-step change $e^{h\lambda}$ of the exact solution when $h\lambda$ is small, and that the other one is small leading to rapid decay of the corresponding mode.

(d) Find the value of $h\lambda$ at which stability is lost as h increases with λ real and negative.

1 (a) 2-step AB is $y_{j+2} = y_{j+1} + h\left(\frac{3}{2}f(y_{j+1}) - \frac{1}{2}f(y_j)\right)$ (for autonomous)

$= y_{j+1} + h\lambda\left(\frac{3}{2}y_{j+1} - \frac{1}{2}y_j\right)$ for $f(y) = \lambda y$.

(b) $$\begin{bmatrix} y_{j+1} \\ y_{j+2} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{1}{2}h\lambda & 1+\frac{3}{2}h\lambda \end{bmatrix}}_{M} \begin{bmatrix} y_j \\ y_{j+1} \end{bmatrix}$$

$$\det(M - \lambda z) = \det\begin{bmatrix} -z & 1 \\ -\frac{1}{2}h\lambda & 1+\frac{3}{2}h\lambda - z \end{bmatrix} = \underbrace{(-z)\left(1+\frac{3}{2}h\lambda - z\right) + \frac{1}{2}h\lambda}_{\text{char poly}}$$

$$\boxed{\begin{gathered} z^2 - \left(1+\frac{3}{2}h\lambda\right)z + \frac{1}{2}h\lambda = 0 \\ z = \frac{\left(1+\frac{3}{2}h\lambda\right) \pm \sqrt{\left(1+\frac{3}{2}h\lambda\right)^2 - 2h\lambda}}{2} \end{gathered}}$$

(c) $$= \frac{\left(1+\frac{3}{2}h\lambda\right) \pm \sqrt{1 + \underbrace{3h\lambda - 2h\lambda}_{h\lambda} + \frac{9}{4}(h\lambda)^2}}{2}$$

$$= \frac{1}{2}\left[\left(1+\frac{3}{2}h\lambda\right) \pm \left\{\underbrace{1 + \frac{1}{2}\left(h\lambda + \frac{9}{4}(h\lambda)^2\right) - \frac{1}{8}\left(h\lambda + \frac{9}{4}(h\lambda)^2\right)^2 + O(h^3)}_{\text{Taylor expanded } \sqrt{1+\varepsilon}}\right\}\right]$$

$$= \begin{cases} \frac{1}{2}\left[1 + \frac{3}{2}h\lambda + 1 + \frac{1}{2}h\lambda + \frac{9}{8}(h\lambda)^2 - \frac{1}{8}(h\lambda)^2 + O(h^3)\right] \\ \frac{1}{2}\left[1 + \frac{3}{2}h\lambda - 1 - \frac{1}{2}h\lambda + O((h\lambda)^2)\right] \end{cases}$$

$$= \begin{cases} 1 + h\lambda + \frac{1}{2}(h\lambda)^2 + O(h^3) = e^{h\lambda} + O(h^3) \\ \frac{h\lambda}{2} + O((h\lambda)^2) \xrightarrow[(h\lambda)\to 0]{} 0 \end{cases}$$

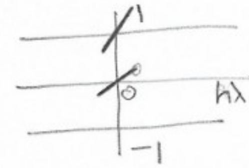
QED.

One eigenvalue approximates e^{h lambda}, the other is close to zero.

1(d) Char poly is $z^2 - \left(1 + \frac{3}{2}h\lambda\right)z + \frac{1}{2}h\lambda = 0$.

From part (c), we saw for small $h\lambda$

$$z_\pm \approx \begin{cases} 1 + h\lambda \\ \frac{h\lambda}{2} \end{cases} :$$



Want to know where $|z_+|$ or $|z_-|$ first attains value 1 as $h\lambda$ is decreased from 0.

If $\underline{z_\pm \text{ stay real}}$, then must attain $|z| = 1$ at $z = \pm 1$.

We can check for these easily:

$z = +1$: $\quad 1^2 - 1 - \frac{3}{2}h\lambda + \frac{1}{2}h\lambda = h\lambda = 0 \rightarrow h\lambda = 0$ only.

$z = -1$: $\quad (-1)^2 + 1 + \frac{3}{2}h\lambda + \frac{1}{2}h\lambda = 2 + 2h\lambda = 0$

$$\rightarrow \boxed{h\lambda = -1 \text{ only}}$$

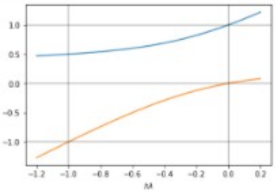
and can conclude stability lost at $\boxed{h\lambda = -1}$.

To guarantee $z_\pm$ do stay real, could show that discriminant does not change sign on $h\lambda \in (-1, 0)$.

I'll make a quick plot as sanity check.

```python
import numpy as np
import matplotlib.pyplot as plt
def axhlines(vals):
    for val in vals: plt.axhline(val,color='k',alpha=.3)
def axvlines(vals):
    for val in vals: plt.axvline(val,color='k',alpha=.3)

```

```python
hlambda = np.linspace(-1.2,0.2,40)
a = 1
b = -(1+3/2*hlambda)
c = 1/2*hlambda
zp = (-b + np.sqrt(b**2 - 4*a*c))/2/a
zm = (-b - np.sqrt(b**2 - 4*a*c))/2/a
plt.plot( hlambda,zp )
plt.plot( hlambda,zm )
axhlines([-1,0,1])
axvlines([-1,0])
plt.xlabel('$h\lambda$');
```



2.

Ack 7.10.16

$$y_{j+1} = y_j + h\Phi(t_j, y_j, h)$$

$$\Phi(t_j, y_j, h) = \frac{1}{4} f(t_j, y_j) + \alpha f\left(t_j + \frac{h}{4}, y_j + \beta h f(t_j, y_j)\right)$$

(a) Consistent iff $\Phi(t_j, y_j, 0) = f(t_j, y_j)$

Here $\Phi(t_j, y_j, 0) = \frac{1}{4} f(t_j, y_j) + \alpha f(t_j\ y_j)$

$$= \left(\frac{1}{4} + \alpha\right) f(t_j, y_j)$$

Thus for consistency, $\boxed{\alpha = \frac{3}{4}}$.

(b)

$$y_{j+1} = y_j + h\cdot\left(\frac{1}{4} f + \frac{3}{4}\left[f + f_t \cdot \frac{h}{4} + f_y \beta h f + O(h^2)\right]\right)$$

$$= y_j + hf + h^2\left(\frac{3f_t}{16} + \frac{3\beta}{4} f_y \cdot f\right) + O(h^3)$$

while

$$y(t_j + h) = y_j + hy' + \frac{h^2}{2} y'' + O(h^3)$$

$$= y_j + hf + \frac{h^2}{2}\left(f_t + f_y \cdot f\right) + O(h^3)$$

(i) The method is order $p = 1$ regardless of the value of β.

(ii) No value of β can make the method of order $p = 2$ (for general f) because $\frac{3}{16} \neq \frac{1}{2}$ regardless of β.

(For an <u>autonomus</u> system, $f_t \equiv 0$, and the method is of order 2 if $\frac{3}{4}\beta = \frac{1}{2}$, ie. $\beta = \frac{2}{3}$.)

3. LMF Venn diagram

(i) Place the following LMFs in the Venn diagram above:

a. full 2-step (7.85 in textbook) This is consistent but not stable, hence not convergent.

b. $y_{j+1} = y_j/10$ This is stable but not consistent (even c0 not zero).

c. $y_{j+1} = y_j + 2hf_j$ This is stable (one eigenvalue = 1 for f=0) but not consistent (c1=−1: order of accuacy 0).

d. Euler This is stable and consistent (c0=c1=0, c2≠0: order of accuacy 1).

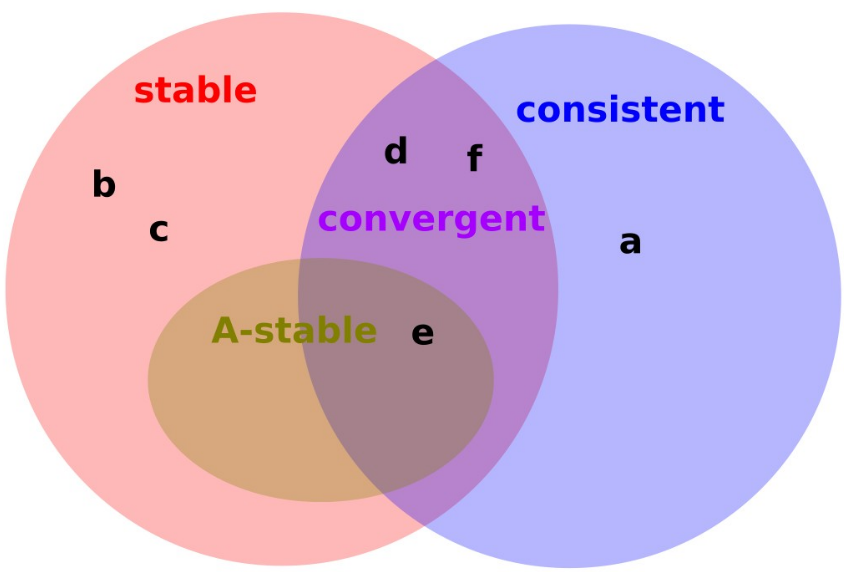
e. Backward Euler: $y_{j+1} = y_j + hf(t_{j+1}, y_{j+1})$ Stable and consistent (c0=c1=0, c2≠0: order of accuacy 1), and A-stable.

f. Adams-Bashforth methods Stable and consistent with order increasing with k. There are no explicit A-stable methods.

(ii) lightly shade the region of the diagram comprising methods that are convergent. Shaded purple.

(ii) Give an example of a method that is neither consistent nor stable.

y_{j+1} = 20 y_j



4. An implicit method for stability on a stiff system

(a) Implement the trapeziodal method (see p423) to solve the stiff system,

$$Y' = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} -y \\ x \\ -41(z - 2x^2 - y^2) \end{bmatrix}.$$

with the initial condition $Y(0) = [3, 0, 18]^T$, going to t=10 with h=.05. Use a fixed number of Newton iterations. Plot x,y,z vs. t in 3 separate subplots.

Print the value of x,y,z at t=10 so I can compare your results with mine precisely.

(b) Compare with using Euler's method with the same step size, and superimpose the results on the plot from part (a).

Again, print the value of x,y,z at t=10, if you get that far.

Also upload your code to UBlearns.

Hints

You need to create functions like:

```
def f(t,y):
    ...

def Df(t,y):
    ...
```

To solve the matrix-vector equation Ax=b:

```
np.linalg.solve(A, b)
```

To get the identity matrix:

```
np.eye(3)
```

Calculate the Newton step $s = -Dg^{-1}g$ by solving $Dg\ s = -g$ for s.

Don't confuse the jacobian of the function g whose root you want to find with the jacobian of f (the right hand side of the differential equation), though the former is expressed simply in terms of the latter.

4. A nonlinear stiff system causes trouble for Euler, not for Trapezoid.

imports

```python
from matplotlib import rcdefaults
rcdefaults()  # restore default matplotlib rc parameters
%config InlineBackend.figure_format='retina'
import seaborn as sns  # wrapper for matplotlib that provides prettier styles and more
import matplotlib.pyplot as plt  # use matplotlib functionality directly
%matplotlib inline
sns.set()

import numpy as np
```

functions to execute Euler and trapezoid steps

```python
def euler_step(t,yj,fDf,h):
    y = np.array(yj)
    fj = fDf(t,yj)[0]
    return y + h*fj

def trapezoid_step(t,yj,fDf,h):
    y = np.array(yj)  # use current y value as initial guess for future value
    fj = fDf(t,yj)[0] # function of diff eq
    newtonsteps = 1
    for k in range(newtonsteps):
        f,Df = fDf(t,y) # function and jacobian of diff eq
        g = y - yj - h/2*(fDf(t,y)[0] + fj)  # value of function g that's supposed to be zero
        Dg = np.eye(len(y)) - h/2*Df          # jacobian of g
        s = -np.linalg.solve(Dg,g)  # Newton step
        y = y + s
    return y
```

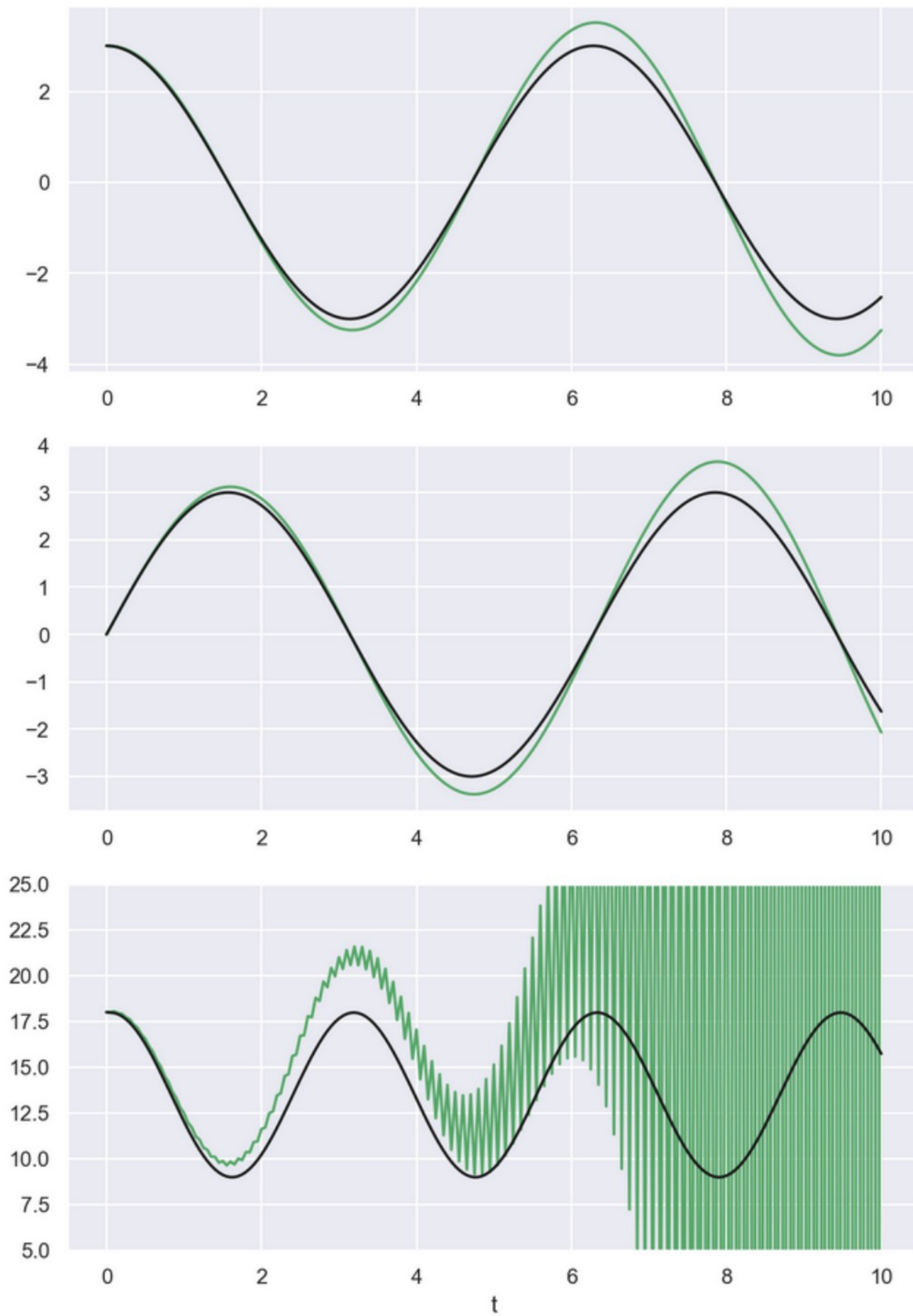
just taking 1 Newton step per timestep

A nonlinear stiff system

```python
# another stiff example
import numpy as np

def fDf(t,Y):
    x,y,z = Y
    a = 41.
    b = 2
    f = np.array([-y,x, -a*(z-b*x**2-y**2) ])
    Df = np.array([[0,-1,0],
                   [1,0,0],
                   [2*a*b*x,2*a*b*y,-a]])
    return f,Df

def ic(t):
    return np.array([3,0,18.])

t0 = 0.0;
t1 = 10
M = 200
h = (t1-t0)/M
print(h)
ya = np.empty((3,M+1))

fig,ax = plt.subplots(3,1,figsize=(8,12))
for method,color in zip([euler_step,trapezoid_step],'gk'):
    y = ic(t0)
    ya[:,0] = y
    for j in range(M):
        t = t0 + h*j
        y = method(t,y,fDf,h)
        ya[:,j+1] = y
        #break
    ta = np.linspace(t0,t1,M+1)
    for i in range(3):
        ax[i].plot(ta,ya[i,:],'-',color=color)
ax[2].set_ylim(5,25); ax[2].set_xlabel('t');
```

a function to compute both f and Df

Results: Euler in green, trapezoid in black.



Trapezoid looks good. Euler is unstable.